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ABSTRACT

The purpose of a seminar attended by mathematics educators from the United States and Japan was to explore ways that use of technology in teaching problem solving can improve student learning. The following presentations were made: (1) "A Report of U.S.-Japan Cross-cultural Research on Students' Problem-Solving Behavior," (T. Miwa and T. Fujii); "Results of U.S.-Japan Cross-cultural Research on Students' Problem-Solving Behaviors" (J. Becker); (3) "An Overview of Computer Use for Mathematical Problem Solving in Japanese Schools" (T. Miwa); (4) "Calculators, Computers, and Algebra" (J. Fey); (5) "Using Construction Programs in the Teaching of Geometry" (J. Choate); (6) "On the use of Software in Mathematics Classrooms in Japan" (Y. Sugiyama, T. Kaji, and Y. Shimizu); (7) "Graphing in the K-12 Curriculum: The Impact of the Graphing Calculator" (F. Demana); (8) "A Study of Problem Solving with Cabri-Geometry in Secondary School Mathematics" (N. Nohda); (9) "Some Examples of Mathematics Instructional Software in Japanese Classrooms" (S. Sakitani and S. Iida); (10) Software Demonstrations by the Japanese Delegation (Y. Morimoto); (11) "Where Do Functions Come From? Data Analysis in Secondary Mathematics" (D. Teague); (12) "The Use of Computers in Schools: Some Findings From National and International Surveys" (T. Sawada); (13) "The Use of Computers in Schools: The State of Using Computers in Mathematics Classrooms in Japan" (K. Kumagai); (14) "Presentation of the Hawaii Geometry Learning Project to the U.S. Japan Seminar" (W. G. Martin) (handout provided at seminar); (15) "Computer Use in Teaching Mathematical Problem Solving--Pre-Service Teacher Education at Yokohama National University" (Y. Hashimoto); (16) "Graphical Reasoning for New Approaches to Topics in Mathematics" (S. Dugdale); (17) Software Demonstration by U.S. Delegation (J. Choate and D. Teague); (18) "Mathematical Visualization in Problem Solving Facilitated by Computers" (J. Wilson); (19) "Mathematical Education in a High-Technological Information Oriented Society--What Should It Be?" (T. Uetake); (20) Summary of Seminar (T. Miwa and J. Becker). A delegates list, opening and closing statements, proposed topics for further study, descriptions of software, and a list of supplementary papers are included. (MDH)

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Proceedings of the U.S.–Japan Seminar on Computer Use in School Mathematics

Editors

Jerry P. Becker
Southern Illinois University at Carbondale

Tatsuro Miwa
Institute of Applied Optics

SE 053 383

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PROCEEDINGS OF THE U.S.-JAPAN SEMINAR ON COMPUTER USE IN SCHOOL MATHEMATICS

Edited by

Jerry P. Becker
Department of Curriculum and Instruction
Southern Illinois University at Carbondale
Carbondale, Illinois USA

Tatsuro Miwa
Institute of Applied Optics
Tokyo, Japan

July 15-19, 1991

at

The East-West Center, Honolulu, Hawaii

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At the East-West Center

Insets: Yasuhiro Morimoto, Nancy Whitman, Neil Pateman (Missing: John Haig)
Back Row: James Wilson, Howard Wilson, Jonathan Choate, Daniel Teague, Toshio Sawada, James Fey, Gary Martin, Frank Demana, Shin'ji Iida
Middle Row: Yoshinori Shimizu, Suzanne Damarin, Yoshihiko Hashimoto, Toshiko Kaji, Sharon Dugdale, Joseph Zilliox, JoAnn Kaida, Seiichi Kaida, Toshiakira Fujii
Front Row: Tom Ferrio, Shin'ya Sakitani, Jerry Becker, Tatsuro Miwa, Yoshishige Sugiyama, Tsuneo Uetake, Nobuhiko Nohda, Koichi Kumagai

AMERICAN DELEGATES

Name	**Address**
Jerry P. Becker (Co-Organizer)	Department of Curriculum and Instruction Southern Illinois University at Carbondale Carbondale, IL 62901-4610
Sharon S. Dugdale	Division of Education University of California at Davis Davis, CA 95616-8579
Jonathan Choate	Groton School P.O. Box 991 Groton, MA 01450
Daniel J. Teague	North Carolina School of Science and Mathematics 1219 Broad Street Durham, NC 27705
James T. Fey	Department of Mathematics University of Maryland College Park, MD 20742
Franklin D. Demana	Department of Mathematics Ohio State University 231 West 18th Avenue Columbus, OH 43210
James W. Wilson	Department of Mathematics Education 105 Aderhold Hall University of Georgia Athens, GA 30602
Observers	
Nancy C. Whitman	Department of Curriculum and Instruction University of Hawaii 1776 University Avenue Honolulu, Hawaii 96822
Suzanne K. Damarin	Educational Policy and Leadership 122 Ramseyer Hall Ohio State University 29 West Woodruff Avenue Columbus, OH 43210
Howard L. Wilson	Department of Mathematics Oregon State University Corvallis, OR 97331-4605

Joseph T. Zilliox
Department of Curriculum and Instruction
WA2-223
University of Hawaii
1776 University Avenue
Honolulu, Hawaii 96822

JoAnn T. Kaida
St. Andrew's Priory School
224 Queen Emma Square
Honolulu, Hawaii 96813

Tom Ferrio
Texas Instruments
P.O. Box 650311
MS 3946
Dallas, TX 75265

Gary Martin
Curriculum Research and Development Group
University of Hawaii
1776 University Avenue
Honolulu, Hawaii 96822

Neil Pateman
Department of Curriculum and Instruction
WA2-223
University of Hawaii
1776 University Avenue
Honolulu, Hawaii 96822

JAPANESE DELEGATES

Name	Address
Tatsuro Miwa (Co-organizer)	The Institute for Applied Optics Bld. Kandanishikimachi 203 3-20 Kandanishikimachi, Chiyoda-ku Tokyo 101, JAPAN
Tsuneo Uetake	Asia University 5-24-10, Sakai, Musashinoshi Tokyo 180, JAPAN
Toshio Sawada	National Institute for Educational Research 6-5-22 Shimomeguro, Meguro-ku Tokyo 153, JAPAN
Yoshishige Sugiyama	Tokyo Gakugei University 4-1-1, Nukui Kita-machi, Koganei-shi Tokyo 184, JAPAN
Nobuhiko Nohda	The Institute of Education University of Tsukuba 1-1-1 Tennodai, Tsukuba-shi Ibaraki 305, JAPAN
Yoshihiko Hashimoto	Yokohama National University 156 Tokiwadai, Hodogaya-ku Yokohama-shi 240, JAPAN
Shin'ya Sakitani	Hyogo University of Education Yashiromachi-kume, Katoogun Hyogo 673-14, JAPAN
Toshiakira Fujii	Yamanashi University 4-4-37 Takeda, Kofu Yamanashi 400, JAPAN
Toshiko Kaji	Tokyo Gakugei University 4-1-1, Nukui Kita-machi, Koganei-shi Tokyo 184, JAPAN
Yasuhiro Morimoto	NEC Corporation 2-7-17, Shiba, Minato-ku Tokyo 105, JAPAN
Shin'ji Iida	Fukuoka University of Education 729, Akama, Munakatashi Fukuoka 811-41, JAPAN

Koichi Kumagai	Joetsu University of Education 1, Yamayashikimachi, Joetsu Niigata 943, JAPAN
Yoshinori Shimizu	Tokyo Gakugei University 4-1-1, Nukui Kita-machi, Koganei-shi Tokyo 184, JAPAN

TRANSLATORS

Seiichi Kaida

St. Andrews Priory School
224 Queen Emma Square
Honolulu, Hawaii 96813

John Haig

East Asian Languages and Literature
Moore 356
University of Hawaii
Honolulu, Hawaii 96822

PREFACE

Jerry Becker and Tatsuro Miwa, Co-Organizers

These are the Proceedings of the U.S.-Japan Seminar on Computer Use in School Mathematics held at the East-West Center in Honolulu, Hawaii July 15-19, 1991. The Seminar and these Proceedings mark the importance placed on technology in school mathematical education in both the United States and Japan during the decade of the 1990's. They also represent a natural extension of cross-cultural collaboration in research in problem solving, following an earlier Seminar in July 1986. In this spirit, the first two papers represent a transition from the collaborative research that has been underway to this Seminar.

We believe, along with all the delegates, that the Seminar was a success. Interesting, scholarly papers and discussions filled the Seminar Agenda. It was also an enjoyable event held in the superb facilities of the East-West Center with the beautiful Japanese Garden in the background. Not only did the participants find the Seminar valuable, but the event marked the mutual and increasing interest by mathematics educators from both counties in extending communication, exchange and cross-cultural collaboration in research.

We want to extend our heartiest appreciation to all the delegates who, through their paper presentations and discussion, accounted for so much of the quality interaction during the Seminar. We need to also express appreciation to the National Science Foundation (NSF) and the Japan Society For the Promotion of Science (JSPS) which, through the U.S.-Japan Cooperative Science Program, made this Seminar possible. In particular, Mr. Warren Thompson and Ms. Patricia Tsuchitani have our thanks for their important role in the Seminar's success.

No bi-national seminar such as this can be successful without competent translators. In this respect, the Seminar was exceedingly fortunate to have Mr. Seiichi Kaida and Dr. John Haig as translators. Not only were they highly knowledgeable about the intricacies of translating between English and Japanese, but they were friendly, amiable individuals who worked patiently and tirelessly to smooth communication during Seminar sessions and social activities. To both we extend our profound appreciation.

As mentioned above, the facilities of the East-West Center are superb and certainly they were ideal for our Seminar. For providing comfortable meeting arrangements, an excellent technical setup, and a staff of friendly and supportive individuals, we need to convey our deep appreciation to Mr. James McMahon, the Logistics Officer of the East-West Center. Through Mr. McMahon's support and patience, our Seminar was helped to success. To members of his staff goes our sincere thanks: Cathy Yano, Kathleen Clark, and Norma Heen.

This Seminar was an important one and it is important to note that it is a natural extension of the collaborative research which followed the earlier Seminar in 1986. The first phase focused on students' problem solving behaviors and with the current growing use of technology (computers and calculators) in both countries, researchers on both sides agreed that another Seminar to explore its use in teaching and problem solving was needed. All agreed that such a Seminar would be useful, as well as timely, and it was decided to seek support by submitting proposals simultaneously to the NSF and JSPS. These proposals were reviewed on both sides and recommended for support. There ensued preparation on both sides covering a time period of 1-2 years, culminating in this Seminar at the East-West Center.

We also express our appreciation to Nancy Whitman, Gary Martin, Joe Zilliox and Neil Pateman of the University of Hawaii. They assisted the Co-Organizers in a multitude of ways; in particular, they provided some of the hardware needed for the Seminar and participated in the sessions. Gary Martin also hosted a very useful and informative visit to the Hawaii Geometry Learning Project. To Ms. Joan Griffin goes our heartfelt appreciation for her enormous energy and friendly competence in transcribing the discussion tapes and in assisting in getting all the manuscripts into final form. We also thank Mary Jane Schaaf, Karen Stotlar and Lois Cornett for their help in revising manuscripts after editing. Ms. Ming Wang provided computer expertise in transporting all the papers into Microsoft Works - to her goes our enormous gratitude. Others assisted in various ways in finalizing this report, though any mistakes are the responsibility of the Editors.

Various companies provided hardware and/or donations of software and other materials for demonstration at the Seminar. We acknowledge their generosity below, and express our profound gratitude since software demonstrations were a crucial part of the Seminar:

NEC/Japan (Mr.Yasuhiro Morimoto)
Sunburst Communications (Stacey Yaruss)
Key Curriculum Press (Steve Rasmussen)
Computer-Intensive Algebra (Drs. James Fey and Kathleen Heid)
Function Probe (Dr. Jere Comfrey)
DERIVE
IBM (Mathematics Exploration Kit)
Mathcad
Houghton-Mifflin (Geometry Grapher)
Texas Instruments (Tom Ferrio)

It is our earnest hope that these Proceedings will be of interest to mathematics educators in both countries, as well as to others who share our desire to advance the cause of an improved mathematics education for children and students at all school levels. Further, we are hopeful that the cross-cultural research that has been underway may continue and that the results will contribute to a better understanding of the potential that computer use holds for educating students in mathematics.

Jerry P. Becker　　　　　　　　　　Tatsuro Miwa

August 1992

Special Note: The American delegation was pleased to host the Seminar in Honolulu. We wish to include a special acknowledgement to members of the Japanese delegation, all of whose members prepared and presented their papers in English in excellent fashion. This represented a significant effort on their part, an effort for which we are deeply appreciative.

Jerry P. Becker

SEMINAR PURPOSES

There continues to be considerable interest and a large number of activities in mathematics education in both Japan and the United States. Mathematics educators in both countries are exploring ways in which student learning can be improved in all areas of school mathematics. But a present area of great concern and an area on which mathematics educators of both countries are focusing their attention is the use of technology in teaching problem solving. Accordingly, this is the focus for the present Seminar and subsequent research.

During the research subsequent to the 1986 Seminar involving American and Japanese mathematics educators and continuing right up to 1992, a great and mutual interest was expressed in continuing to bring mathematics educators on both sides together to improve communication and to propose and conduct further research. This joint U.S.-Japan Seminar seemed like an excellent manner by which to do this. The main purposes of the Seminar were set as follows:

1. to examine the present state of technology use in school mathematics in the United States and Japan,
2. to explore classroom practices using technology in the United States and Japan,
3. to examine existing data and research concerning technology use in the two countries,
4. to see demonstrations of software used in each country,
5. to discuss software development and the philosophy underlying its use,
6. to make plans for cross-cultural research in technology use in problem solving in the U.S. and Japan.

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Institute of Applied Optics

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Southern Illinois University at Carbondale

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OPENING CEREMONY

Monday, July 15, 1991

Miwa: Good morning Ladies and Gentlemen.

On behalf of the Japanese delegation, I convey our most sincere greetings to members of the American delegation. It is our great pleasure to come back here again, to meet old friends and to become acquainted with new friends. Just five years ago, in July 1986, Professor Becker and I organized the U.S.-Japan Seminar on Mathematical Problem Solving held here at the East-West Center. Subsequently, in the years 1988 and 1989 U.S. and Japanese colleagues collaborated in research on students' problem solving behaviors. Both were supported by the National Science Foundation (NSF) and the Japan Society for the Promotion of Science (JSPS). This Seminar is surely an extension of that Seminar and research into a study of urgent issues which is needed in the mathematical education community today; that is, to foster students' mathematical problem solving ability with use of computers.

Needless to say, problem solving has been and is now a major focus of school mathematics in both the United States and Japan. It is well known that emphasis is placed on teaching mathematical problem solving in classrooms in both countries. On the other hand, use of technology, in particular, of computers, is an urgent task in the mathematics educational community not only in the U.S. and Japan, but all over the world.

This Seminar addresses the use of computers in school mathematics from a cross-cultural viewpoint. By computer we mean mainly the microcomputer, but also the hand-held calculator. The purposes of the Seminar are to integrate recent studies on computer use in school mathematics and in mathematical problem solving in both countries, to investigate a framework for further study and to set a research agenda which seems both possible and necessary. We in the Japanese delegation sincerely hope that the U.S. researchers will give us kind advice since the U.S. is a pioneer and forerunner in this area. In order that there be a large success in the Seminar, I ask for the cooperation of all of you.

At this point, I would like to offer my sincere thanks to members of the U.S. delegation. In particular, to Professor Becker who has been very kind and careful in handling arrangements for the Seminar so perfectly. Also, I offer my sincere thanks to the East-West Center which has provided the Seminar rooms, technical support and various facilities, and especially to Mr. McMahon who has done so much to make our conference comfortable. In thinking about the Seminar, with its focus on cross-cultural matters, the East-West Center is, perhaps, the most appropriate place for our meeting. In addition, I am very grateful to the NEC corporation which transported Japanese computers to Hawaii which will enable us to demonstrate Japanese software. Finally, I want to express thanks to our translators Mr. Seiichi Kaida and Professor John Haig. Our hope is that they will assist in lowering the language barrier and thereby contribute towards the success of the Seminar. Thank you very much.

Becker: Thank you, Professor Miwa, for the greetings of your delegation and very generous remarks. All of my American colleagues join me in expressing what a great honor it is for us to have such a distinguished group of Japanese mathematics educators in our country and in this beautiful center to participate in this Seminar. On behalf of the entire U.S. delegation, we welcome you to the United States and to the East-West Center. Also, we express our appreciation to Mr. James McMahon who, of course, has been generously working with Professor Miwa and me for many months in making the arrangements for the Seminar.

I would like to take a couple of minutes to give a brief history of how it is that we have come together to discuss the use of technology in teaching mathematics at the school level. Professor Miwa has already touched on this a little. But, back in the early 1980s when I was attending a meeting in Tokyo, Professor Shigeru Shimada and Professor Miwa approached me and raised the question of a possible U.S.-Japan seminar on mathematics education. Sometime after that we became acquainted with the U.S.-Japan Cooperative Science Program, organized under auspices of the National Science Foundation in the United States and the Japan Society for the Promotion of Science in Japan. We were subsequently able to acquire funding support from them and we convened delegations, as Professor Miwa has said, in July 1986 here at the East-West Center. And that represented the first phase of our collaboration. At that Seminar, we decided that we would publish the Proceedings of the Seminar and that we would also engage in cross-national research on students' problem solving behaviors. That work has been under way for some time now with support from the National Science Foundation and the Japan Society for the Promotion of Science. In fact, Professor Miwa, his colleagues and I had some discussions some time ago in which we thought we would like to begin this Seminar by providing a transition from the last Seminar, and subsequent collaborative research, to this Seminar. As Professor Miwa has said, we can see this Seminar as a natural extension of joint, collaborative work that has been going on with U.S. and

Japanese mathematics educators for several years. In the United States there has been a great deal of emphasis on the use of technology in teaching mathematics and problem solving. For example, recent and highly authoritative books published by NCTM (i.e., the Curriculum and Evaluation Standards (1989) and the Professional Standards For Teaching Mathematics (1991)), address the question of technology use in school mathematics. Similarly, in Japan there is a new revision of the national curriculum in mathematics (i.e., Mathematics Program in Japan (1989)) which recommends integration of technology into school mathematics teaching. So, technology is now very important in both of our information- oriented societies and, therefore, in teaching mathematics at the school level.

Once again, on behalf of the American delegation, we welcome our Japanese colleagues to the Seminar and look forward to continuing our work together.

Miwa: Thank you. For this occasion we brought Japanese textbooks, from the elementary through the upper secondary school levels, and the translation of the new Course of Study which details the content taught in our educational system. In addition, we have brought copies of Basic Facts and Figures About the Educational System in Japan. It is our honor to present these materials to the U.S. delegation. I want to mention, also, that one series of textbooks, for the secondary level, has been translated into English by the University of Chicago School Mathematics Project (UCSMP).

Becker: I am happy to accept these materials on behalf of our delegation and, on behalf of the entire U.S. delegation, please accept our sincere appreciation for these gifts. The book Basic Facts and Figures About the Educational System in Japan contains a lot of very useful and interesting information about education in Japan. The Mathematics Program in Japan is the new syllabus, for implementation beginning in 1992. We appreciate these new materials very much. Reciprocating, the U.S. delegation has brought to the Seminar a collection of software that can be used in teaching mathematics at the school level, for developing mathematical thinking and problem solving abilities. During the working group session on Thursday, we will demonstrate much, but probably not all, of the software. My colleagues join me in presenting these materials to you, Professor Miwa, and the Japanese delegation. In addition, we have one other little gift which we would like to hand out. For each of you, we have a coffee mug from the National Council of Teachers of Mathematics which we'll hand to you in just a little while.

Miwa: Thank you very much. On behalf of the Japanese delegation, we are grateful to the U.S. delegation for the software, which we will study and which will be very useful. We also thank you for the coffee mugs which will serve as a reminder and souvenir of this Seminar. Thank you very much.

A REPORT OF U.S.-JAPAN CROSS-CULTURAL RESEARCH ON STUDENTS' PROBLEM SOLVING BEHAVIORS

Tatsuro Miwa
Institute for Applied Optics
Tokyo, Japan

Toshiakira Fujii
University of Yamanashi
Yamanashi, Japan

Introduction

In this report, we would like to present an analysis of Japanese data concerning the five "Common Survey" problems given as part of the U.S.-Japan Collaborative Research on Students' Problem Solving Behaviors. The aim of the Common Survey is to examine commonality and differences of strategies and difficulties between U.S. and Japanese students at a variety of grade levels when they solve non-routine problems.

Students were asked to write not only answers, but also the ways or approaches of finding answers. In addition, they were asked not to erase anything written down, but to draw a line through anything they felt was in error. Through analysis of students' writing we hope to elucidate and identify the nature of students' problem solving behaviors. In addition, we administered a questionnaire concerning students' attitudes and abilities about mathematics and towards the problems administered to students. These results will reveal affective aspects of U.S. and Japanese students' mathematical problem solving behaviors.

Generally, analysis of the results of problem solving tends to focus on the correctness of answers. One characteristic of our common survey is to analyze and compare students' ways of thinking between the two countries through examination of their scripts. In this report, we considered correct and incorrect answers alternatively in pairs in terms of the way of solving that produced them. Then we related the rate of correct answers with the approaches chosen.

The results reported here are for Japanese data. We look forward to comparing the Japanese and U.S. data, which appears at the moment in the draft papers by U.S. researchers.

Methodology

Problems Given

Researchers of both countries proposed several problems as candidates for the Common Survey Problems during 1988-89; then the problems were tried out in the spring, 1989. The problems were revised, then finally administered about the same time in each country's school year during 1989-90. In addition to the problems, student questionnaires concerning their affective aspect of mathematical problem solving were developed, along with a teacher questionnaire to gather information about the schools, characteristics of the classes, teachers' opinions of student attitudes towards solving the problems, and teachers' reactions to the problems. The survey was administered according to instructions which were also developed among the researchers. We made decisions to collect the data at the 4th, 6th, 8th and 11th grades, and two problems were administered for each grade with one of the two problems overlapping two grades, as listed below:

Problems	Grade Levels
1 Marble arrangement	4
2 Number of matchsticks	4, 6
3 Number of marbles	6, 8
4 Arithmogon	8, 11
5 Areas of squares	11

(Note 1) The five problems are shown in the Appendix.

(Note 2) In Japan, grades 4 and 6 are in elementary school, grade 8 is in lower secondary school and grade 11 is in upper secondary school, the second year of lower and upper secondary school, respectively. In Japan, elementary and lower secondary education are compulsory and free. Although upper secondary education is not compulsory, its enrollment is now about 94% of the age group.

Subjects in Japan

Subjects were one classroom of students at the grade level of a school in five prefectures: Ibaraki, Yamanashi, Kanagawa, Aichi and Hiroshima. For selection of schools, no statistical methods were applied. Therefore, we cannot say the results represent an average achievement of students in the country in a statistical sense, but rather the results are case studies to find out features in mathematical problem solving behaviors in terms of commonality and differences. In Japan, numbers of subjects and gender were as follows:

Grade	4			6			8			11		
	B	G	T	B	G	T	B	G	T	B	G	T
Number	84	88	172	91	91	182	96	93	189	135	95	234

Grand Total: Boys 406, Girls 371, Total 777

In this report, although subjects consist of boys and girls nearly equally, we are not going to analyze the data in terms of sex differences in achievement.

Administration

Survey tests were administered according to the instruction of one problem for fifteen minutes each; that is, the survey would be within one class period, including the student questionnaire. The teacher let students solve the first problem for the first fifteen minutes and the second problem for the next fifteen minutes. And after solving the two problems, the teacher asked students to respond to the questionnaire for five minutes. According to the instructions, the teachers were asked not to answer any questions from subjects concerning the content of problems; e.g., meaning of the problems, how to start with the problem, etc..

Marble Arrangement*

This problem is for 4th grade students in the elementary school.

The total number of students' responses was 990, and 95% of responses were correct. The mean number of responses per student was 5.8, and 5.9 for boys and 5.6 for girls, respectively. The minimum number of responses by an individual student was 1 and maximum number was 15. The highest percentage of the number of the responses (44%) was six; this may be associated with the number of solution spaces provided in a worksheet.

The results are examined from the two perspectives: one is the ways of solving the problem, (i.e., solution strategy) and the other is mode of explanation (i.e., the manner in which the students justified their answers).

The Mode of Explanation

The students' modes of explanation were identified by taking individuals into consideration. If an individual student used at least one pictorial explanation in his/her responses, we coded him/her as "Have" Figure Explanation. If a student did not draw anything in all problem figures provided, we coded him/her as "Not have". The Words Explanation mode was coded in the same way. The words explanation mode includes words as verbal explanation and/or mathematical expression. In total, 75% of students used both figure and words explanation, and almost all students (97%) used words explanation.

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* Data in this section are taken from Nagasaki(1990) pp. 27-44, (1991) pp.37-52.

Table 1 The Mode of Explanation in Marble Arrangement

		Figure	Explanation	
		Have	Not have	Total
Words	Have	75%	22%	97%
Explanation	Not have	3	0	3
Total		78	22	100

The words explanations consisted of verbal explanations and/or mathematical expressions. These explanations are subdivided with respect to arithmetic operations such as counting, adding, and multiplying. Finally, these were classified into five categories as follows:

1 verbal / counting: count one by one, count by lining, count from the top.
2 verbal / addition: sum up, by adding the marbles.
3 verbal / multiplication: take them into groups, count by fives.
4 mathematical expression / addition: 1+3+5+7+5+3+1, 4+3+4+3+4+3+4.
5 mathematical expression / multiplication: 5x5, 4x4+3x3, 3x8=24 24+1.

In order to analyze students' priorities in the use of words explanation, we made the order of above five categories according to a mathematical point of view as follows: $1<2<3<4<5$. That is, mathematical expression using multiplication as an arithmetic operation is considered the highest mathematical value. Then the modes of explanation were ordered and identified according to the order. For instance, if a student shows 1, 3, 4 in responses, we identify the mode of explanation as 4, which is the highest order among 1, 3 and 4. The following table shows students' priorities in ways in words explanation.

Table 2 Ways in Words Explanation : Students' Priorities

Verbal Explanation			Mathematical Expression	
Counting	Addition	Multiplication	Addition	Multiplication
3%	0%	19%	15%	60%

(Total number of students : 172)

Sixty percent of Japanese 4th graders used multiplication as a mathematical expression in at least one of their responses. The percentage of 60 seems quite high indeed. Multiplication is frequently used by Japanese students.

Ways of Solution

Ways of solving the problem were classified into five categories:

A Enumeration.
B Grouping by number (multiple of 2, 3 etc.)
C Grouping by direction (vertical, horizontal etc.)
D Grouping by figure (enclosing, symmetry etc.)
E Transforming into a figure to apply multiplication (displacement)

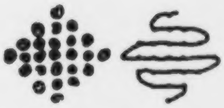
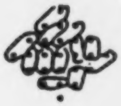
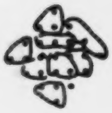
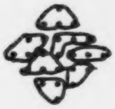
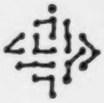
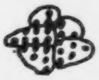
Category A is that students enumerate marbles presumably without considering the configuration in the marble arrangement. Categories B, C and D are responses based upon pattern or structure which students found in the arrangement of the marbles, and category E is that students change the structure of the pattern. These were identified in figurative explanation, of which the total number was 603. The numbers of responses and of students classified into above categories are shown below:

Table 3 Ways of Solution of Marble Arrangement (Figure Explanations)

		# of Response	# of Student	Re./Stu.
A	Enumeration	60	41	1.5
B	Grouping by number	241	99	2.4
C	Grouping by direction	242	91	2.7
D	Grouping by figure	52	44	1.2
E	Transforming into a figure	8	5	1.6

Table 3 shows that over half of the students used categories B and/or C, while less than one fourth of students used A and/or D. Students tended to change their ways of solution within categories B and C by focusing on the number and directions, respectively. In order to examine responses in more detail, we classify them into the following 38 subcategories, as illustrated in Table 4.

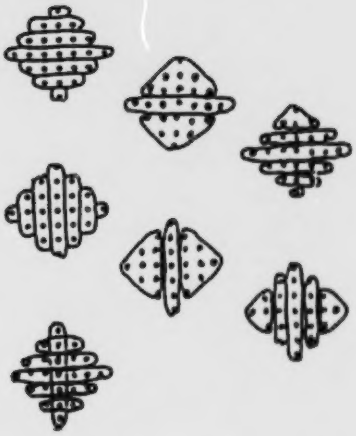
Table 4 Subcategories for Ways of Solution (Figure Explanations)

	Number of Responses	Number of Students	Examples
A Enumeration			
A-1 Count one by one	60	41	
B Grouping by number			
B-1 Making multiples of the same number			
B-1-2 Multiple of 2	31	31	
B-1-3 Multiple of 3	35	33	
B-1-4 Multiple of 4	39	37	
B-1-5 Multiple of 5	57	51	
B-1-6 Multiple of 6	16	12	
B-1-7 Multiple of 7	9	9	
B-1-8 Multiple of 8	5	5	
B-1-9 Multiple of 9	4	4	
B-1-10 Multiple of 10	33	32	
B-1-11 Multiple of 11	2	2	
B-1-12 Multiple of 12	4	4	
B-1-13 Miscellaneous	6	4	

C Grouping by directions

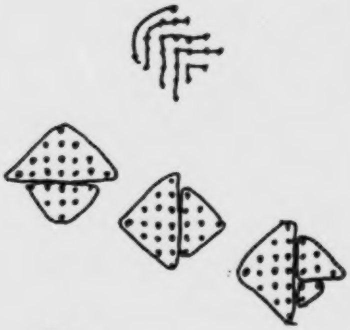
C-1 Horizontal - vertical

C-1-1	Horizontal	35	28
C-1-2	Accumulated hori.	12	12
C-1-3	Transformed hori.	14	12
C-1-4	Vertical	36	30
C-1-5	Accumulated vert.	14	12
C-1-6	Transformed vert.	11	6
C-1-7	Mixed hori.& vert.	2	2
C-1-8	Connecting hori.& vert.	0	0



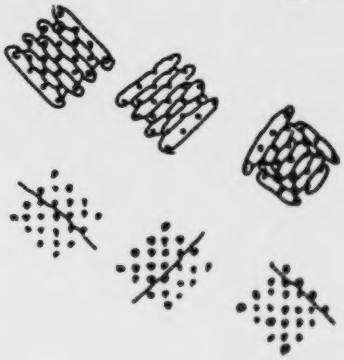
C-2 Top-bottom & Left-right

C-2-1	Partition by Top-bottom	44	23
C-2-2	Partition by Left-right	12	5
C-2-3	Miscellaneous	9	5

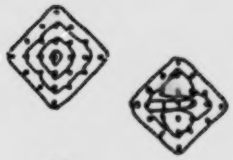


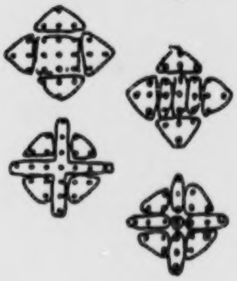
C-3 Oblique

C-3-1	Right down	17	17
C-3-2	Left down	23	22
C-3-3	Mixed	13	10
C-3-4	Divided and right down	0	0
C-3-5	Divided and left down	0	0
C-3-6	Transformed oblique	0	0



D	Grouping by figures		
D-1	Enclosing		
D-1-1	Purely enclosing	14	14
D-1-2	Transformed enclosing	4	4
D-2	Central axis		
D-2-1	Central region	4	4
D-2-2	Transformed central re.	12	10
D-2-3	Symmetry	9	9
D-2-4	Transformed symmetry	9	8
E	Transforming into a figure		
E-1	Displacement	8	5
E-2	Supplement	0	0





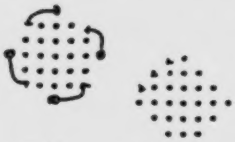
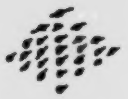
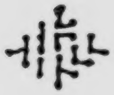
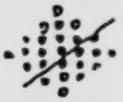
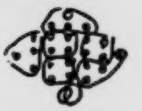
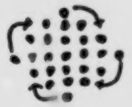


Table 4 shows that many students tend to group marbles in various ways. This tendency seems to relate to their tendencies to use a multiplication mode in justifying their answers. Taking into account individuals to analyze students' ways of finding structure in the situation, we could identify five categories "a" to "e" shown in the Table 5. We value these five categories as follows : a<b<c<d<e. That is, if a student uses three categories, say "a", "c" and "d", then we identify him/her in category "d". Rates of students in five categories are shown in the Table 5.

Table 5 Ways of Finding Structure in the Figure

a	Marking each marbles one by one	3%	
b	Drawing a continuous line or lines connecting marbles	5	
c	Dividing the structure by drawing a line	1	
d	Grouping by some subsets with the same number of marbles	66	
e	Displacement or adding some new marbles	3	

In Table 5 we see that 66% of students grouped the figure by some subsets. Grouping the marbles into sets with the same number may naturally lead to realize a multiplicative structure of the figure, and then students could represent it by a multiplication mode. If they use the category "b": drawing a continuous line or lines connecting marbles, they would use the enumeration and/or addition mode because marbles connected with lines may only lead to insight for an addition structure. That Japanese students use a multiplication mode of explanation, rather than an addition mode, may be associated with this fact.

Number of Matchsticks*

Ways of Solution

We identified the following five categories of students' ways of solutions whether they lead to correct answers or not, taking underlying mathematical ideas into consideration:

A Consider square as a unit figure, and eliminate overlapping edges.
Ex. 4 x 8 – 7 = 25, 4 x 8 = 32

B Consider figure] or [as a unit figure, and add a terminal edge or a square. ( ] looks like a Japanese Katakana character, pronounced "ko".)
Ex. 3 x 8 + 1 = 25, 3 x 8 + 4 = 25, 3 x 7 + 1 = 22

C Calculate horizontal and vertical edges respectively.
Ex. 2 x 8 + 7 + 2 = 25, 2 x 8 + 8 = 24

D Draw figures and count one by one.

E Others.

Results for Grade 4 and Grade 6 students are illustrated in the Table 6. The fourth graders tended to use the method of type A, type B and type D. In type A, fourth graders resulted more often in an incorrect answer. With detailed observation, it becomes clear that fourth graders were mainly indifferent to overlapping. In other words, the majority of them ignored overlapping or shared sides and used a mathematical expression: 4x8=32. In addition, more fourth graders used type D than sixth graders. In this case, not a few made errors in counting. In sixth graders, most used method was type B. This is because they tried to avoid involvement of overlapping situation. Instead, they tried to interpret the figure as the Katakana character of コ. This fact implies that the better strategy could reach the higher rate of correct answer.

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* Data in this section are taken from Fujii (1990) pp. 47-64, (1991) pp. 53-72

Table 6 Ways of Solution of Matchsticks

Grade	4			6		
	Total	Correct	Incorrect	Total	Correct	Incorrect
A Square as a unit figure	23%	1%	23%	16%	5%	11%
B Figure as a unit figure	28	24	4	43	38	5
C Calculate hori. and vert.	8	6	2	6	6	1
D Draw figures and count	28	22	6	10	9	1
E Other miscellaneous	12	4	8	25	11	14
No Answer	1	-	1	1	-	1
Total	100	57	43	100	67	33

We examined the mode of explanation for solution and identified ten categories for the manner in which each response was justified by the student. Results are shown in the Table 7.

Table 7 shows that the mode of explanation involving the mathematical expressions increased from fourth graders (64%) to sixth graders (79%). This result seems to be consistent with the results analyzed in the Marble Arrangement Problem. Japanese students tend to use mathematical expressions when they try to justify their answers.

Table 7 Mode of Explanation in Number of Matchsticks

	Grade 4		Grade 6	
Math. expression only	20%		36%	
Math. expression, Figure	12	64%	17	79%
Math. expression, Verbal explanation	20		19	
Math. expression, Figure, Verbal expl.	12		7	
Verbal explanation only	13		7	
Verbal, Figure	11		5	
Figure only	11		8	
Table only	0		1	
Answer only	1		1	
No Answer	0		1	

Problems Made up by Students

We focus on the first problem students produced, because it was the problem students made up just after getting a solution to the given problem and would reflect what they had thought out in the solving process. Problems made up by students are classified into six categories as shown in Table 8.

Type 1 problems are basically produced by changing the condition given in the original problem. Type 2 problems are the converse of the original one. A typical example is a problem asking the number of figures which can be constructed by a given number of matchsticks. Type 3 problems are different in the materials and also in situation from the original one.

We will examine Type 1 problems in more detail to identify students' thinking in producing new problems. First, we consider the conditions involved in the original problem, which is shown below.

Squares[3] are made by using matchsticks[5] as shown in the picture[2]. When the number of squares is eight[4], how many matchsticks[1] are used?"

Table 8 Problems Made up by Students

Type 1 Problems asking the number of edges		63%
	* with overlapping	51
	* without overlapping	6
	* undecided with regard to overlapping	5
Type 2 Converse problem		5%
	* with overlapping	3
	* without overlapping	1
	* undecided with regard to overlapping	1
Type 3 Problems on four rules		10%
	* multiplication	5
	* division	2
	* addition and subtraction	2
	* mixed	1

Type 4 Others		7%
	* construction	1
	* Ueki-zan (planting tree probl)	1
	* problem of area	1
	* others	4
Type 5 Write other solutions of given problem		7%
Type 6 Unidentified & No writing		8%

We can grasp its mathematical and non-mathematical constituents as follows:

[1] It is a problem asking the number of edges.
[2] There are overlapping sides in constructing unit figures.
[3] A unit figure is a square.
[4] The number of units is eight (eight in a row).
[5] The materials are matchsticks.

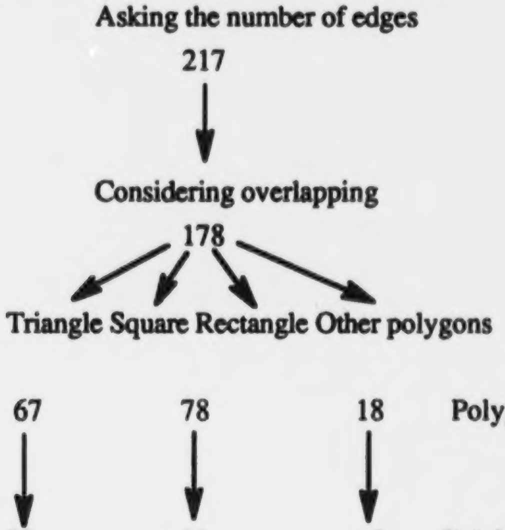
Problems newly made up by students are those in which constituents are changed from the original ones.

1 Ask the number of edges --- ask the number of vertices, of particular edges.
2 Have overlappings --- change the way of overlapping, without overlapping,
whether have overlapping or not.
3 Square --- other polygons (e.g., triangle, hexagon, rectangle), polyhedra.
4 Eight (eight in a row) --- nine or more in a row, m row & n column,
circular arrangement, spatial arrangement.
5 Matchsticks --- pencils, rulers.

Two hundred and seventeen problems which were made up as the first problem of Type 1 by students are structured as illustrated in the Figure 1 below.

Figure 1 reveals that on construction of figures in students' made problems from the most were square (78), triangle (67) and rectangle (18), on arrangement the most is one row n column, n>8. That is, students tend to make up new problems changing from eight squares in the given problem to n squares or n triangles, n>8.

Figure 1 Number of Type 1 Problems Made up by Students


Asking the number of edges
217
Considering overlapping
178
Triangle Square Rectangle Other polygons

		67	78	18	Polygons 13
1xn(n<8)	(Matchst.)	10	11	6	Polyhedra 2
	(Others)	0	6	2	
1x8	(Matchst.)	3	3	1	(Details are
	(Others)	0	2	0	not described.)
1xn(n>8)	(Matchst.)	31	33	5	
	(Others)	8	11	3	
2xn(n<8)	(Matchst.)	0	5	0	
	(Others)	1	0	0	
2x8	(Matchst.)	1	1	0	
	(Others)	0	0	0	
mxn(m>2	(Matchst.)	2	5	1	
n>8)	(Others)	0	0	0	
Pyramid	(Matchst.)	4	0	0	
	(Others)	1	0	0	
Others	(Matchst.)	4	1	0	
(Others)		2	0	0	

Number of Marbles*

This problem is for the 6th grade students in the elementary school and 8th grade students in the lower secondary school.

Rate of Correct Answers between Questions

In this problem, rate of correct answer between questions is as follows:

Table 9 Rates of Correct Answer in Number of Marbies

Grade	6 (N=141)#	8 (N=188)
Question #1	90%	96%
Question #2	28	61
Question #3	24	37

Note: (# In this problem, performance of students of sixth grade in Aichi Prefecture was excellent by far, from that of others, and their results were eliminated from the analysis.)

Table 9 shows that both sixth and eighth graders were successful in Question #1. But the rate of correct answer decreased from Question #1 to #3. The sixth graders' rate of correct answer decreased dramatically in Question #2, compared with eighth graders' results. In contrast, the eighth graders' rate of correct answer decreased in Question #3, where they needed to use literal symbols in algebra. This reveals that eighth graders had difficulty in making up mathematical expressions involving variables.

In order to examine the above results from a different viewpoint, we classified students according to the correctness of answers in three sequential questions as illustrated in the following:

1 Those whose answers were correct in all Questions #1, #2 and #3.
2 Those whose answers were correct in Questions #1 and #2 but not in #3.
3 Those whose answers were correct only in Question #1.
4 Other.

Rates of respective categories are shown below:

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*Data in this section taken from Ishida, J. (1990)pp. 65-86, (1991) pp. 73-96.

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Table 10 Rates of Correct Answers in Sequential Questions

Grade	6	8
1	21%	34%
2	7	27
3	59	33
4	13	7

Table 10 shows again that, for sixth graders, there was a large gap between the fourth term (Qu. #1) and sixteenth term (Qu. #2) in generalizing the pattern in Question #1.

Ways of Solution

From now on, we will focus on students who got correct answers for Question #1, because the rate of correct answers in #1 is 90% or above for students.

Observing students' scripts for this problem, we can identify the following ways of solving the problem:

A Draw a figure of marbles and count one by one.

B Apply a rule that in each term there are four rows,
in which the number of marbles increase one by one from the top row.
Ex. 4+5+6+7 for Qu. #1

C Apply a rule that when the number of term increases by one, the number of marbles increases by four.
1 Calculate the increment and add it to the first term value 10.
Ex. 4x3 + 10 for Qu. #1.
2 Multiply 4 by the number of terms, and add to six.
This is the same way as C1, but it is easy to generalize.
Ex. 4x4 + 6 for Qu. #1.
3 Calculate the increment and add it to the term other than the first.
This is a slight transformation of C1.
Ex. 4x2 + 14 for Qu. #1, as 14 is a value for the second term.

C' Add 4 to the immediately prior term.
Ex. 18+4 for Qu. #1

D Apply the formula for the area of a trapezoid or parallelogram.

E Make up a table and find the pattern.

F Others.

G Errors and no answer

Errors in Type G are classified as follows:

S Answer only increment, i.e., 4xn for the nth term.

R Use Proportional strategy.

Ex. The first term is 10, then the 16th is 10x16.

Ex. The fourth term is 22, then the 16th is 22x4.

H No Answer.

Results of the analysis of ways of solution are shown in Table 11 for 6th graders and Table 12 for 8th graders.

Table 11 Ways of Solution for Questions #1, #2 and #3: Grade 6
(Those who got correct answers for #1 only)

	A	B	C				C'	D	E	F	S	R	W
			1	2	3	Total							
Qu. #1 Corr.	13%	18%	10%	6%	1%	17%	39%	2%	0%	11%			
Qu. #2 Corr.	1%	12%	13%	3%	3%	19%	2%	0%	0%	2%			
Qu. #2 Inco.	1	6	12	0	2	14	2	0	1	11	11%	6%	13%
Qu. #2 Total	2	17	24	3	5	32	5	0	1	13	11	6	13
Qu. #3 Corr.	0%	9%	13%	3%	2%	18%	0%	0%	0%	1%			
Qu. #3 Inco.	1	6	13	0	2	15	1	0	0	8	12%	3%	28%
Qu. #3 Total	1	15	26	3	4	33	1	0	0	9	12	3	28

In applying the solution method used at the fourth term to the sixteenth term and then hundredth or nth term, ways of solution or mathematical expressions at the fourth term would largely influence students' achievement. In other words, if students solved Question #1 with the solution method which had potential for generalizing the pattern, they would have the possibility of getting answers in Question #2 and #3. In fact, nearly all those of C2 type in Question #2 (4x16+6) got the correct answer for Question #3. But these students were not many.

Viewing Tables 11 and 12, we can find that ways of solution used for question #1 were from the most C' (39%), B (18%), and A (13%) for sixth graders, and C' (30%), C (19%), A (15%), B (13%) for eighth graders. Unfortunately, the most often used method did not have potential for generalizing the pattern and for getting correct answers in Question #2 and #3.

The rate of category A decreases dramatically from Qu. #1 to Qu.#2 and #3 for both grade levels. This shows that drawing a figure for the 16th and 100th or nth is not easy or impossible and students belonging to category A changed their methods of solution.

Table 12 Ways of Solution for Question #1,#2 and #3: Grade 8
(Those who got correct answers for #1 only)

	A	B	C				C'	D	E	F	S	R	W
				1	2	3	Total						
Qu. #1 Corr.	15%	13%	19%	9%	0%	28%	30%	3%	5%	7%			
Qu. #2 Corr.	1%	18%	24%	14%	2%	40%	3%	3%	2%	0%			
Qu. #2 Inco.	0	5	10	0	0	10	4	0	4	2	2%	3%	5%
Qu. #2 Total	1	23	34	14	2	50	7	3	6	2	2	3	5
Qu. #3 Corr.	0%	4%	15%	15%	0%	30%	0%	2%	0%	0%	0%	0%	0%
Qu. #3 Inco.	0	3	22	0	1	23	0	1	0	0	4	0	30
Qu .#3 Total	0	7	37	15	1	53	0	3	0	0	4	0	30

For sixth graders, while the rate of category B is constant from Qu. #1 to #2 and #3, the rate of C is about twice that of Qu. #1 for #2 and #3 and that of C' decreases radically and is one tenth of Qu. #1 for #2 and only 1% for #3. In addition, the rate of S is not negligible for Qu. #2 and #3 and category W is especially large for this grade. This reveals that many sixth graders had difficulty in making up mathematical expressions that could generalize from the pattern they had found at the fourth term. These facts may explain the reason why the rate of correct answers were decreasing dramatically in question #2 and #3.

For eighth graders, the rate of category B increases between Qu. #1 and #2 but decreases between #2 and #3. On the other hand, the percentage of C increases from Qu. #1 to #2 and #3 and becomes over 50%, but the percentage of C' decreases to one fourth of Qu. #1 for Qu. #2 and vanishes for #3. Most noticeable is rate the W, especially for Qu. #3, which is greater than that of sixth graders.

Finally, we note that some students assumed the marble arrangement to be a geometric figure like a trapezoid and applied a formula of measuring the area. They made a connection between arithmetic and geometry. It is to be recommended in mathematics.

Arithmogon*

This problem is for 8th grade students in the lower secondary school and 11th grade students in the upper secondary school.

Rate of Correct and Incorrect Answers between Problems

Problem 1 (Triangular arithmogon) has a unique solution, while problem 2 (Square arithmogon) has infinitely many solutions, i.e., any integer is possible to be substituted for, say, top left circle and gives a solution. Therefore, in problem 2, we distinguished those who got the correct answer, i.e., infinitely many solutions, from those who got one correct solution with no indication of more than one or infinitely many solutions.

Responses of students are shown in the Table 13 below. In the Table 13 we see that the rate of correct responses is 39% and 90% of eighth and eleventh graders, respectively, for Problem 1. For Problem 2, the rate of correct responses surprisingly decreased to only 1% for both eighth and eleventh graders. The percent of "Got One Answer" is 38% for 8th graders and 24% for 11th graders. The percent of "Incorrect answer" increased from 21% for 8th graders to 95% for 11th graders, which is not expected and surprised us.

Table 13 Results of Responses for Arithmogon

	Problem 1		Problem 2	
Grade	8	11	8	11
Correct Answer	39%	90%	1%	1%
Got One Answer	-	-	38	24
Incorrect Answer	52	8	21	55
No Answer	9	2	40	20

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*Data in this section are taken from Senuma,(1990)pp. 87-101,(1991)pp. 97-114

Ways of Solution

As the ways of solving problems 1 and 2, we identified the following seven ways for problem 1 and three for the problem 2.

* Problem 1 (Triangular Arithmogon)

1 Simultaneous linear equations with three variables.

Ex. $x + y = 63$

$y + z = 21$

$z + x = 38$

2 Simultaneous linear equations with two variables.

Ex. $x + y = 63$

$y + (38 - x) = 21$

3 Linear equation with one variable

Ex. $(63 - x) + (38 - x) = 21$

4 Sum of three numbers in squares (63, 38 and 21) is divided by 2, or discovery of a numerical structure in the problem.

5 Systematic substitution.

Ex. Systematically substitute 10, 20, 30 and so on for the top circle.

6 Trial and error and/or guess.

Ex. First substitute an arbitrary number, say 25, for the top circle, then adjust the numbers in the circles.

7 Others

* Problem 2 (Square Arithmogon)

8 Simultaneous linear equations with four variables

Ex. $x + y = 23$

$y + z = 52$

$z + u = 47$

$u + x = 18$

9 Trial and error and/or guess

Ex. First substitute an arbitrary number, say 1, for the top left circle, then adjust the numbers in the circles.

10 Others

Tables 14 and 15 show results for ways of solution for Problem 1 and for Problem 2, respectively.

In Problem 1, 40% of eighth graders used only the trial and error/guess, and half of them got correct answers. On the other hand, 10% of 8th graders got correct answers using linear equations, among which the most common is simultaneous equations with three variables. Fifty-nine percent of eleventh graders got the correct answer only by simultaneous equations with 3 variables, and in total 80% of 11th graders got the correct answer using at least an algebraic method, i.e., linear equations.

In Problem 2, 35% of eighth graders are the type of "Got One Answer" by only trial and error/guess. On the other hand, 50% of eleventh graders could not get the correct answer by simultaneous equations with 4 variables.

Table 14 Ways of Solution for Problem 1

Grade	8			11		
	Correct	Wrong	Total	Correct	Wrong	Total
1 Si. Eq. 3 vari.	7%	7%	14%	59%	4%	63%
2 Si. Eq. 2 vari.	2	2	4	1	-	1
3 Li. Eq.	1	1	1	2	-	2
4 Sum. div. by 2	2	-	2	-	-	-
5 Sys. Sub.	3	1	4	2	-	2
6 Tri. Err.	21	19	40	5	1	6
7 Others	1	20	21	1	2	3
1 and 2	-	-	-	2	-	2
1 and 3	-	-	-	3	-	3
1 and 4	-	-	-	6	-	6
1 and 5	1	-	1	0	-	0
1 and 6	2	2	4	5	-	5
1 and 7	1	-	1	2	0	2
2 and 3	-	-	-	0	-	0

2 and 6	-	1	1	-	-	-
3 and 4	-	-	-	1	-	1
4 and 5	1	-	1	0	-	0
4 and 6	-	-	-	0	-	0
6 and 7	-	1	1	-	-	-
1, 2, 4 and 7	-	-	-	0	-	0

Table 15 Ways of Solution for Problem 2

Grade	8				11			
	Corr.	One.A	Wrong	Total	Corr.	One.A	Wrong	Total
8 Si. Eq. 4 vari.	0%	0%	7%	7%	0%	5%	50%	56%
9 Tri. Err.	1	35	5	42	0	10	1	12
10 Others	-	1	9	10	0	1	3	5
8 and 9	-	2	-	2	-	8	0	8
9 and 10	-	1	-	1	-	-	-	-

Many students were brought to their wits' end and stopped solving halfway through the problem. Thus, "trial and error/ guess" led 8th graders to a correct answer or "Got One Answer"; on the other hand, "simultaneous equations" led 11th graders to incorrect answers.

The reason why the rate of eleventh graders in Problem 2 was less than for eighth graders would be the fact that they applied the method, which was successful for Problem 1, to Problem 2. When they made up simultaneous equations with four variables in Problem 2, the equations were not linearly independent and could not have a unique solution. They would have difficulty and trouble with it and reach the error. On the other hand, eighth graders who would use trial and error and/or guess method could get a correct answer or at least one answer as a candidate.

Areas of Squares*

This problem is for 11th grade in the upper secondary school.

Ways of Solution

Examining students' scripts, we identified and classified ways of solution for this problem into following five categories:

A1 Applying algebraic expression of degree 2.

Ex. Length of AB and AP are expressed as a and x, respectively.
$y = x^2 + (a-x)^2 = 2(x-a/2)^2 + a^2/2$. y is a minimum when $x = a/2$.
P is midpoint of AB.

A2 Computation of numerical value.

Ex. Let AB=10. AP=5, then sum of areas is 50; AP=6, then sum is 52; AP=7 then sum is 58 and so on. AP is a minimum when AP=5.

A3 Drawing figures.
A4 Verbal explanation.
A5 Others including no answer.

We added another viewpoint regarding correctness of answers, as follows:

B1 Correct answer. (Conclusion "Midpoint of AB" is indicated.)
B2 Incorrect or no answer.

Further, we paid attention to the reasoning of reaching a conclusion, and classified students belonging to category B1 "correct answer" into the following six subcategories:

B11 General and deductive reasoning.
B12 General and deductive reasoning, but interrupted halfway.
B13 Concrete and inductive reasoning.
B14 Inappropriate reasoning.
B15 Wrong interpretation of the problem.
B16 Without reason.

We did the same for those belonging to category B2 "Incorrect or no answer".

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*Data in this section are taken from T. Ishida, (1990) pp. 103-120, (1991) pp. 115-132

Table 16 Ways of Solution for Areas of Squares

	A1	A2	A3	A4	A5	Total
B 1 (Correct)	21%	25%	17%	13%	8%	83%
B 11	15	-	-	-	-	15
B 12	5	-	-	-	-	5
B 13	-	21	17	4	-	42
B 14	1	4	-	9	1	14
B 15	-	-	0	0	-	1
B 16	-	-	-	-	7	7
B 2 (Incor.)	7	1	3	2	3	17
B 22	4	-	-	-	-	4
B 23	-	1	3	0	-	4
B 24	3	0	-	1	-	4
B 25	0	-	-	0	-	1
B 26	-	-	-	-	3	3
Total	28	26	20	15	12	100

Results are shown in Table 16. The rate of the correct conclusion, that is, the midpoint of AB was indicated, is 83%. But, as we demanded both correctness of conclusion and valid reasoning, only 15% of students satisfied both conditions.

We see that ways of solutions of students are from the most A1 Applying algebraic expression of degree 2, A2 Computation of numerical values and A3 Drawing figures. As to reasoning, 46% of students based their solutions on concrete and inductive reasoning, and 24% of them on general and deductive reasoning. It is unexpected that so many 11th graders used inductive reasoning, in spite of the fact that they had studied mathematical reasoning for many years.

Problems Made up by Students

Problems made up by students were classified into two categories, Valid and Invalid, and they were subdivided into the following five subcategories:

Valid Problems

1 ordinary problems in mathematics.
2 ordinary problems in mathematics, though solution is self-evident.
3 problems with insufficient conditions and insolvable immediately.

Invalid Problems

4 problems incapable to identify intention and unable to call problems.
5 problems interrupted halfway.

The number of valid problems made up by students was 380, or 92% of the total of 414 problems produced, and invalid ones were 8%. The mean number of valid problems made up per student is 1.6

Valid problems are classified into the following four types, taking features of problems into consideration:

a Similar to the original problem.
The problem has the following features which are characteristic of the given original problem.
  i) A variable point exists.
  ii) At least two figures exist.
  iii) Variables concerning the figure are identified.
  iv) A problem asks for the position of a variable point when some conditions are imposed on the variable.
  Problems of this type are those made up changing some constituents of the original problem.

b Converse of the original problem.
A typical problem is the one that asks the value of variable when the position of the variable point is determined.

c Quasi-similar to the original problem.
The problem has two or three features of the original problem. Features are those described in category "a".

d Dissimilar to the original problem.
The problem has one or no features of the original problem. That is, problems of this type have little relevance to the original one.

Classification of valid problems into the above categories are shown below.

Table 17 Type of Problems Made up by Students

Type	a	b	c	d
Rate of students	79%	4%	11%	6%

Seventy nine percent of students were of Type "a", and those of the others were small. It is noticeable that students of Type "d" (problems dissimilar to original one) are only 6%. That is, students usually made up problems changing some constituents of given original problem slightly. More detailed analysis shows that in Type "a" problems, the most frequent change is from a square to a triangle, and the next is from minimum to maximum.

Results of Questionnaire*

The survey included a questionnaire consisting of seven items to gather information about students' attitudes toward mathematics and their opinions of the administered problems. We concentrate on the results of the following two items. It is noted that results below exclude rate of "neutral" answers.

Like or Dislike of Mathematics

First item was "Do you like Mathematics?" Results are shown below:

Table 18 Responses for "Do you like Mathematics?"

Grade	4	6	8	11
Like	42%	34%	24%	18%
Dislike	6	21	27	30

The rate of Japanese students who like mathematics decreases the higher the grade level. On the contrary, the number of students who dislike mathematics increases dramatically the higher the grade level. At grade four, only 6% dislike mathematics; but in grade eleven 30% of students dislike it. This Japanese situation is very serious.

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*Data in this section are taken from T. Ishida (1991) pp. 23-24

Good or Not at Mathematics

Second item was "Are you good at Mathematics?" Results are shown below:

Table 19 Responses for "Are you good at Mathematics?"

Grade	4	6	8	11
Good	17%	17%	11%	6%
Not good	15	28	44	52

For this item, the results are very similar to the first. Rates of Japanese students who said that they were good at mathematics are very low, at most 17%, and decrease the higher the grade level. In grade eleven, only 6% of the students chose "good" and 52% chose "not good". This Japanese situation is serious. We wonder if students are not confident in doing mathematics.

Final Remarks

1 Features of the Study

Usually an analysis of the results of problem solving in mathematics tends to focus on the rates of correctness or incorrectness of answers. In this study, we tried to avoid that. Instead, we considered correct and incorrect answers alternatively in pairs in terms of the ways of solution that produced them. Then we related the rate of correct answers with the ways of solution or strategies used.

2 Students' Approaches

In making the comparison between the two grades that solved the same problem, the distribution of approaches for solution showed that the sophisticated approach produced the better results. By sophisticated approach, we mean that the larger frequencies of correct answers can be produced and have potential for forming more general solutions. From this point of view, students' mathematical behaviors showed an acceptable tendency; that is, the major approach chosen by students seemed to change from a naive one to a sophisticated one the higher the grade level.

However, the Arithmogons problem is an exception. The eleventh graders were less successful in comparison with eighth graders in solving problem 2. They seemed to rely on or stick to the formal approach, simultaneous equations, and were unable to go back to a naive but powerful one in terms of getting insight. In other words, they were not flexible enough to change the approach from a sophisticated one, that produced the correct answer in problem 1, to a naive one, such as trial and error strategy.

3 Students' Mode of Explanation

The mode of explanation was another aspect to investigate students' problem solving behaviors. In comparison of the results between the grades, we found students tended to use more verbal and mathematical expressions as the grade level goes up. This tendency would reflect everyday activities in mathematics classes in Japan. That is, students are used to communicating their ideas about what they found out, between teacher and students, and between students using mathematical expressions when they had finished the problem given in their classes.

Results also show the tendency to use mathematical expressions involving multiplication in the elementary school. This tendency is even revealed in fourth graders in counting marbles in the figure. Multiplication is introduced to students at second grade in the elementary school in Japan. Therefore, the curriculum makes possible for them to express their ideas in a multiplication mode. In addition, activities at lower grades such as composition and decomposition of numbers may influence the results. Although these activities are usually not involved in representation of operations, there surely could be some readiness activities for later mathematical formulation.

4 Problems Made up by Students

Problems made up by students seem to categorize within a certain range by changing the conditions given in the original problem. In other words, students seem to be unwilling to break through and create new problems. On the other hand, some created converse problems, which have a mathematically rich nature to them.

5 Students' Attitudes toward Mathematics

It is reasonable to believe that students' attitudes toward mathematics are related positively to their academic performance. For Japanese students, in particular, for upper grade students, it is not the case. In fact, they performed well but they expressed feelings that they did not like mathematics and were not good at mathematics, as the questionnaire data indicated. We wonder if they were not confident in doing mathematics. This is a serious problem for Japanese mathematics educators.

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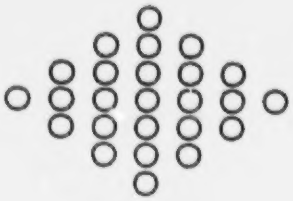
Appendix

Marble arrangement(Grade 4)

How many marbles are there in the picture below?

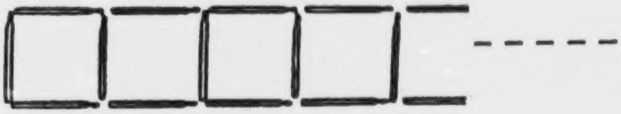


Find the answer in as many different ways as you can.
Write your ways of findings the answer and write your answer.

Note: Following the presentation of the problem, six figures, each with a separate solution space, were provided.

Number of matchsticks (Grade 4, 6)

Squares are made by using matchsticks as shown in the picture. When the number of squares is eight, how many matchsticks are used?



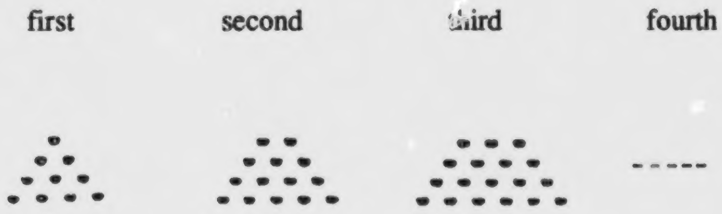
(1) Write your way of solution and the answer.

(2) Now make up your own problems like the one above and write them down.
Make up as many as you can. You do not need to find the answers to your problems.

(3) Choose the problem you think is the best from those you wrote down above,
and write the reason or reasons you think it is the best.
Note: Following the presentation of the problem : question (2), five separate solution spaces were provided for making up problems.

Number of marbles(Grade 6, 8)

Marbles are arranged as follows:



(1) How many marbles are there in the fourth place?

(2) How many marbles are there in the sixteenth place?

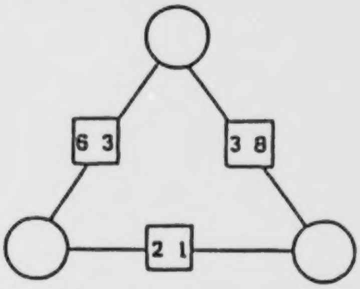
(3) How many marbles are there in the hundredth place?
(for 6th grade)

(4) How many marbles are there in the nth place?
(for 8th grade)

Arithmogon (Grade 8, 11)

Problem 1

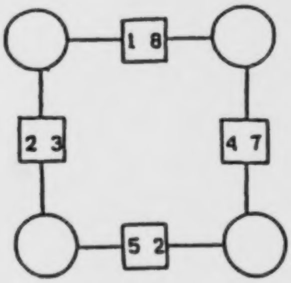
Given a three-sided arithmogon as in the figure below. We put three numbers in the three □ --the number in each □ must equal the sum of the numbers in the two ○ on either side.



Find the numbers for ○ at each corner. The numbers in ○ may be negative numbers. Do not erase anything you write down, just draw a line through anything you feel is in error.
FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN.

Problem 2

Now change to a square (four-sides) arithmogon as in the figure below. The number in each □ must equal the sum of the numbers in the two ○ on either side.
Try to find the number for ○ at each corner.



If you need more space, write on the back of this page.

Areas of squares (Grade 11)

Pick a point P on the line segment AB and make squares : one side of one is AP and one side of the other is PB. Where should the point P be located to satisfy the condition that the sum of the areas of the two squares is a minimum ?

Question 1 : Write a way of solution and the answer to the one above and write them down.

Question 2 : Now make up your own problems like the one above and write them down. Make as many problems as you can.

Discussion of Professor Miwa's and Professor Fujii's Paper

J. Wilson: We have time for some discussion or questions. Does anyone have a question with which they want to start, or a comment?

Becker: Jim, since we started a little late, it will be all right if we take more time for the discussion. Also, Mr. Kaida would like to make a couple comments before we begin.

Kaida: As I mentioned, I am Seiichi Kaida. I will be translating and, during the discussion, please give your name before you ask a question or make a comment.

Teague: I have a question about the problems that were selected for the research. You selected five problems to give to the students. What characteristics or qualities of the problems were important in their selection? Probably you had a number of problems to choose from and you picked these five. Why these five?

Fujii: First of all, they are all non-routine problems. Secondly, students were not familiar with the problems. For instance, we have a national curriculum and if some problems appeared in our textbooks, then the Japanese students would be familiar with them; so, we tried to avoid that situation. Another characteristic of the problems that were selected by the group (that means the Japanese and the American groups together) was that since one of the things we were investigating was the performance of students in developing several or many different ways to solve the same problem, we had to have problems that lend themselves to that. There are not a lot of problems like that around. Now, it happens that the Japanese have had some experience working in the area which we call "open-ended" problem solving and have developed some problems which we included in the research and which worked very well.

Becker: The arithmogons problem, which I learned about from an article in Mathematics Teaching from England (I think McIntosh and Quadling were the authors) was an interesting problem which lent itself to solution in a number of different ways and this was, at least in part, the rationale for including it. Also, it lent itself to application of algebraic techniques, to a certain extent a trial and error approach and to solving the problem by recognizing some structure. There are some very different ways of thinking about the problem, and finding a solution. Some of the ways are included in

Appendix E of my paper. One other comment has to do with students' formulating their own problems. If the structure of the problem is one of the things being looked at, then the extent to which students, once they solve a problem, formulate another problem(s) that has (have) the same structure is important. Once again, there are not a lot of these problems around that have been tried out for research. We had some problems that were proposed by the Japanese and after we all looked at them, we thought they had the characteristics that we wanted.

Demana: On page 2 under 11-graders, it seems that all the categories have essentially the same number of male and female subjects, but not here. Is there any significance to the difference?

Fujii: In Japan, grades one through nine are compulsory education and, therefore, the number of male and female students is just about the same everywhere. Although from tenth grade on only those students who pass the entrance examination can go to upper secondary school, the situation is the same, that is, the number of male and female students is about the same. (In fact, about 95% of students go on to full-time upper secondary school.) But at upper secondary school, classes may be divided into two or three courses according to students' preference. The math-science oriented class may have more male than female in general. Such a situation reflects the different number of male and female subjects.

Choate: I have a comment and a question. When you gave students the problems, did you actually give them matchsticks to use, like in the matchstick problem, for example?

Fujii: No, we didn't.

Choate: There seems to be a method in how the problem was posed for students at different ages - originally a problem was asked for ten matchsticks, at the next level for a hundred, and then the next level **n** matchsticks. Was there a reason for this? Does that reflect what the students are doing at that grade in the schools?

Becker: You made reference to the matchsticks problem, did you mean the marble problem?

Choate: Yes, it was the second marbles problem. Let me simplify my question. There seems in the structure of the problems to be an implicit design of a progression for the students from a particular case to the general case. Is that what you were looking at?

Fujii: Yes, it was intentionally planned that way. Also, the solution to the problem can be found in many different ways.

Becker: I have one other comment about that particular problem. It was used at grades six and eight in Japan, but in the United States we used it at grades 6, 8 and 11 - at grade 11 primarily because the U.S. researchers wondered whether our eleventh grade students in algebra could successfully find the general rule or formula. We felt that, perhaps, our eighth graders could not. And the problem lent itself to answering that question.

Uetake: How much time were students given to solve the problems? Do you think that if students were given more time, they could solve the problems better?

Miwa: The total time was limited to about one hour and, therefore, 15 minutes was allowed for each of two problems. From past experience we thought that 15 minutes was enough. Moreover, even if we were to have given more time, most probably students won't be able to give correct answers in greater frequency.

Becker: We might also mention that after the Japanese and U.S. groups together decided which problems would be used, we then tried the problems out in the U.S. at the grade levels at which we intended to use them. Then the data were analyzed and reported to our Japanese colleagues at a Tsukuba University meeting and, subsequently, we revised the problems. As an example, for the arithmogons problem, the numbers we used in the squares were too easy and led to a positive solution only in the tryout. So, we revised the problem to its present form, in which not all numbers in the circles are positive.

H. Wilson: I am interested in the marble problem and whether or not you analyzed whether a student, in doing the problem in different ways, corrected a mistake. Suppose the student made an error in an early attempt but then found a better or correct answer later. Did that enter your analyses in any way?

Fujii: In this case of the marble problem, 97% of the students gave the correct answer; therefore, your question is not relevant, in this case.

Damarin: I'd like to ask about the mode of solution. In particular, in the two marbles problems and the matchsticks problem, the problems themselves are presented in a figural mode, and not in an algebraic or verbal mode. Were you surprised at the relatively small number of students who used a figural mode in responding to them? Or disappointed, perhaps, in that aspect of solutions?

Fujii: Well, this is my own opinion. A smaller number of students used figures to solve the problem, which we had anticipated. What we focused on was that so many students solved the problems using mathematical expressions. We thought that this was a reflection of their education in mathematics and this was important to us.

Becker: At the eleventh grade level, the problems were stated almost purely in verbal terms, and many students made a figure in their attempt to find a solution to the problem. In one or two other cases, perhaps, the figural nature was a little less prominent.

J. Wilson: I am going to exercise the prerogative of the chair to declare that it is time for a break and ask that we close this discussion. We will have a chance to discuss this data further once Professor Becker's talk is given and the U.S. data are reported. Further questions may be raised then. We will have the break in the Ramon Room and have the full time for informal talk with refreshments.

End of Discussion

RESULTS OF U.S.-JAPAN CROSS-CULTURAL RESEARCH ON STUDENTS' PROBLEM SOLVING BEHAVIORS

Jerry P. Becker
Southern Illinois University at Carbondale

BACKGROUND

The data and results reported here are part of the project on U.S.-Japan Cross-national Research on Students' Problem Solving Behaviors. The research has its origin in the U.S.-Japan Seminar on Mathematical Problem Solving held at the East-West Center in Honolulu, July 14-18, 1986 (Becker and Miwa, 1987).* At that seminar nine U.S. and ten Japanese mathematics educators met to examine the present state of problem solving, explore classroom practices in problem solving, and, in general, to compare the situations in both countries relating to various aspects of problem solving in the classrooms and research (Becker and Miwa, 1987, p. viii).

The last afternoon of the seminar dealt with future communication, exchange of materials and planning cross-national collaborative research. Subsequently, research proposals were submitted, on both sides, to the Division of International Programs of the National Science Foundation (NSF) and the Japan Society For the Promotion of Science (JSPS), respectively, requesting support under the U.S.-Japan Cooperative Science Program. A separate proposal was submitted to the Research in Teaching and Learning Program in the National Science Foundation. The proposals were subsequently funded and the research commenced with a meeting of the U.S. and Japanese groups at the University of Tsukuba in Fall 1988.** The U.S. group made visits to Japanese schools and observed numerous problem solving lessons preliminary to conducting the research (Becker, Silver, Kantowski, Travers, and Wilson, 1990). These visits and the related discussions set the stage for the research which was further broadened and deepened by a visit to the U.S. in the Fall 1989 by the Japanese group which made similar classroom visits followed by further discussions and planning.

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** Members of the groups were: U.S.: Jerry P. Becker (Coordinator), Edward A. Silver, Mary Grace Kantowski, Kenneth J. Travers, and James W. Wilson; Japan: Tatsuro Miwa (Coordinator), Shigeru Shimada, Toshio Sawada, Tadao Ishida, Yoshihiko Hashimoto, Nobuhiko Nohda, Yoshishige Sugiyama, Eizo Nagasaki, Toshiakira Fujii, Shigeo Yoshikawa, Hanako Senuma, Junichi Ishida, Toshiko Kaji, Katsuhiko Shimizu, and Minoru Yoshida.

PROCEDURES AND METHODOLOGY

The U.S. and Japanese groups, hereafter referred to as the group, made decisions to collect problem solving data at the 4th, 6th, 8th and 11th grade school levels, as well as data for preservice elementary and secondary teachers. Problems were selected and administered as follows: one problem at the 4th grade only; one problem at both the 4th and 6th grades; one problem at both the 6th and 8th grades; one problem at both the 8th and 11th grades (in the U.S. case, two problems), and one problem at the 11th grade only. Several of the problems were also used for data collection with preservice teachers.

In addition to the problems, student questionnaires were developed to gather information about students "liking" and "good at" math and their reactions to each of the problems in the research. A teacher questionnaire was also developed to collect information about the schools, teachers' views of their classes, their reactions to the problems and their perceptions of how seriously students worked on the problems. In addition, a set of instructions was developed for use by proctors when the problem booklets were administered.

These materials were developed into preliminary form during the winter, 1988-89 following the Fall, 1988 meeting of the group in Japan. They were "tried out" in the Spring 1989 in classrooms in the Carbondale, IL area. The results were tabulated, reported and discussed at the group's second meeting in Japan in Fall 1989 (Becker, 1989). Subsequently, the materials were revised and finalized for data collection, which occurred at about the same point in each country's school year during 1989-90.

In the formal data collection phase, subjects were given fifteen minutes to work on each of two problems (three at 11th grade) and were asked to write down all their work and to "line out" rather than erase writing. Further, proctors were directed if and when subjects asked questions, to respond by saying "I leave it to your judgment" or "please judge for yourself." In general, students worked on the problems, asking no questions. Each problem was read aloud by the proctor before subjects began work and subjects were stopped promptly after fifteen minutes on each of the two problems (and after ten minutes on the third at the 11th grade). Subjects then filled out the questionnaire during the last five minutes of the class period. Teachers filled out their questionnaire while the problems were being administered. Total time elapsed was forty-five minutes, the usual length of class periods in the schools.

U.S. data for each problem in the study were collected by the five U.S. researchers (Jerry Becker, Kenneth Travers, Edward Silver, Mary Grace Kantowski and James Wilson) in their respective centers around Carbondale (IL), Champaign/Urbana (IL), Pittsburgh (PA), Gainesville (FL), and Athens (GA). In general, for all grade levels, students were attending small or large urban schools. Schools were purposely selected to provide this mix, although the selection of schools and classes within a school was not made in a random manner.

Each researcher analyzed data for one problem which were collected at the four grade levels at each of the centers:

Name of problem	Grade(s)	Researcher
Marble Arrangement	4	Edward Silver
Matchsticks	4,6	Kenneth Travers
Marble Pattern	6,8,11	Mary Grace Kantowski
Arithmogons	8,11	Jerry Becker
Area of Squares	11	James Wilson

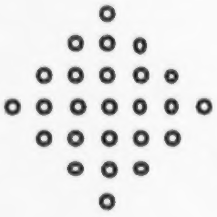
Some of the results for the U.S. sample are reported here, based on drafts of reports by Silver, Travers, Kantowski, Becker and Wilson. The results for the Japanese data are reported in Miwa (1991, 1992). Becker and Silver include comparisons to Japanese results in their reports.

Marble Arrangement Problem/Grade 4

Data for the Marble Arrangement Problem were analyzed and reported by Silver, Leung, and Cai (1991). The problem was presented to subjects as depicted below, followed by nine solution spaces (each with a figure). Ways of students' thinking about the problem are given in Appendix A.

Problem I

How many marbles are there in the picture below?



FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your ways of finding the answer and write your answer.

Silver et al. (1991) used the same basic coding scheme reported in Nagasaki and Yoshikawa (1989), Nagasaki (1990), and Nagasaki (1991/ in Miwa (1991)): solution strategy and mode of explanation, with several different categories for each. Results were reported in four parts: Responses, Solution Strategies, Mode of Explanations, and Questionnaire Responses. There were 151 students (83 boys/68 girls) in the U.S. and 206 students (102 boys/104 girls) in the Japanese samples. A summary of Japanese results is included in their report as well as a comparison of U.S. and Japanese results which are briefly summarized here (pp. 16-20).

A significantly larger percentage (96%) of Japanese students have all correct answers than U.S. students (66%). While there were no significant gender differences in the Japanese results, there were in the U.S., favoring girls. The overall mean numbers of solutions among students giving all correct answers were 7.5 (Japan) and 5.8 (U.S.), which are significantly different. The distributions for solution strategies in the two countries were similar: 90% of students used a "Finding a Structure" strategy at least once, 60% used a strategy of "Enumeration," and less than 5% used a "Change the Structure" strategy. For the two countries, about 33% of U.S. students and 50% of Japanese changed strategies from the first to the fifth response, and when they switched, it was from a primitive to a more advanced one (i.e., from "Enumeration" to "Finding a Structure" to "Change the Structure"). For both countries, students who got correct solutions used "Grouping of Marbles" (36% U.S., 51% Japan). There were more solutions by "Displacement" or "Addition of Marbles" among Japanese than U.S. students, and similarly for "Mathematical Expressions." Finally, there were more "Incomprehensible" solutions among U.S. than Japanese students.

With respect to Mode of Explanation, students in both countries who got all correct answers used both "Figures" and "Words" with the same frequency (about 60%), and more Japanese than U.S. students used "Words." In the U.S., about 20% of students used "Figures" and 20% used "Words," while in Japan less than 5% "Figures," but 36% "Words" only. With respect to verbal and mathematical expressions in Mode of Explanation, Japanese students used mathematical expressions (59% of all responses) while U.S. students had a strong tendency to use verbal expressions (84% of all responses). Further and importantly, Japanese students tended to use multiplication while U.S. students used addition.

Results showed, from students' presentations of solutions, that Japanese students are more used to solving a problem in different ways and are able to effectively communicate their ideas in writing. We do no know the reason(s) for this, nor the reason(s) why Japanese students more commonly use mathematical expressions, while U.S. students use verbal expressions, and Japanese students use multiplication, while U.S. students use addition. Perhaps the reason is the activities in the mathematics curriculum - the teaching and the books.

The authors summarize their findings by noting that students are willing and able to provide multiple approaches to finding the solution to the problem when they are asked to do so and, further, that there is a tendency to use simpler strategies first, and if there is a switch to a different strategy, the tendency is to switch from "Enumeration" to "Find a Structure," or "Change a

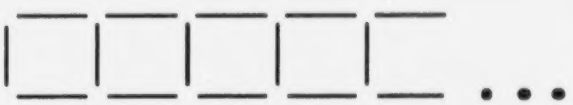
Structure." This move appears to be natural to students in both countries.

Matchsticks Problem/Grades 4 and 6

Data for the Matchsticks problem were analyzed and reported by Travers (1991). The problem was presented to subjects as depicted below, followed by spaces in which students could write their way of solution (part 1), write their own problems (part 2) and write the number of their favorite problem and the associated reason (part 3). Ways of students' thinking about the problem are given in Appendix B.

Problem II

Squares are made by using matchsticks as shown in the picture below. When the number of squares is eight, how many matchsticks are used?



DO NOT ERASE ANYTHING YOU WRITE DOWN; JUST DRAW A LINE THROUGH ANYTHING YOU FEEL IS IN ERROR.

(1) Write a way of solution and the answer to the problem above.

(2) Now make up your own problems like the one above and write them down. Make as many problems as you can. You do not need to find the answers to your problems.

(3) Choose the one problem you think is best from those you wrote down above, and write the number of the problem in the space: __________

Write the reason or reasons you think it is best.

In analyzing the data, Travers used the same basic scheme proposed by Japanese colleagues, with some differences (p. 1), according to the following aspects:

1. Rate of correct answer
2. Methods of solution used to solve the problem
    - A. Breakdown of problems
    - B. Use of drawings
3. Problems made up by students
    - A. Type of problem

    - B. Comparison to Matchsticks problem
        - 1. Object asked for
        - 2. Use of overlap (shared side)
        - 3. Increased dimensions
    - C. Use of illustration
- 4. The problem chosen as best
- 5. Responses to the questionnaires

There were 208 subjects in the U.S. sample: 84 at fourth grade (48 boys/36 girls), 19 at fifth grade (13 boys/6 girls), 105 at sixth grade (45 boys/60 girls).

Results in each section were examined with respect to grade level, sex, and correctness of response for the first aspect. The first three aspects are also examined for relationships among the centers in which students are located, and the relation between method of solution to the given problem and the questionnaire results are examined lastly.

The rates of correct answers were 37% for fourth grade, 58% for fifth, and 52% for sixth. Travers notes that the fifth grade class, which had the highest success rate for all classes at the five centers was above average and small (N=19). Overall, male subjects' rate of correct response was more than ten percentage points higher than that of girls in four of five centers.

Subjects' methods of solutions were categorized as (1) Repetition of Squares (or groups of three matchsticks (e.g.,⊏⊏⊏...), (2) Draw a Picture and/or Count Matchsticks (without using a pattern), and (3) Other (i.e., all other methods). Travers reports that: fourth grade subjects used Drawing/Counting predominantly (70%) and far less frequently used Repetition of Squares (7%); 46% of sixth grade subjects used Drawing/Counting and 30% used Repetition of Squares; 42% of fifth-grade subjects (gifted students) used Repetition of Squares and 37% used Drawing/ Counting. From 21 to 24% of subjects among grade levels 4,5 and 6 used "other" methods of finding their answers. (Note: Travers does not provide examples of these.)

Travers reports no gender differences for methods used to solve the Matchsticks problem for fourth-grade subjects. Both genders used Drawing/Counting most often, and Repetition of Squares less frequently. But the results for sixth grade subjects are different; girls more frequently (37%) than boys (22%) use Repetition of Squares; and boys more frequently than girls use Drawing/ Counting (49% and 43%, respectively) and other strategies (49% and 43%, respectively).

For both fourth and sixth grade subjects, those who got the correct answer more frequently use Counting than Repetition of Squares than subjects whose responses were incorrect. In both categories (correct/incorrect), however, the majority of subjects used Drawing/Counting, except for sixth grade subjects who answered incorrectly. In this group, 44% used Repetition of Squares and 26% used Drawing/Counting - the majority of boys (53%) used Repetition of Squares while girls (35%) used Drawing/Counting. According to Travers, regardless of grade, sex, or correctness, more than 60% of all other groups used Drawing/ Counting. Overall, sixth-grade subjects were more likely to solve the problem using Repetition than fourth-grade subjects, but subjects who use Drawing/Counting were more frequently correct than those who used Repetition of Squares, regardless of sex (perhaps these subjects overlooked the "last" vertical matchstick).

Travers found that students used drawings in solving the problem at all grade levels: 70% for fourth grade, 79% for fifth, and 71% for sixth. Thus, use of drawings appears quite consistent for all grade levels. What is interesting about this is that drawings are unnecessary and subjects can find the answer using a multitude of solution approaches (e.g., 8x3+1=25; 4+7x3=25; 8x2+9=25; 8x2+8+1=25; 6x4+1=25), though drawings can be helpful using these approaches too. What is not clear from the data analysis is whether drawing all eight squares first more or less implies counting to find the answer. Further, Travers's results show that, for fourth grade subjects, 78% of girls and 65% of boys used a drawing, but for sixth-grade subjects the difference was negligible. In addition, whether or not subjects got a correct solution, a majority of both genders used drawings, and 42% of fourth and 63% of sixth grade subjects who used drawings got the correct answer, compared to 24% of fourth and 27% of sixth grade subjects who did not. Drawings appear to be an important crutch for U.S. fourth and sixth grade subjects in solving the Matchsticks Problem.

We turn now to the results for problems formulated by subjects, after they solved or attempted to solve the Matchsticks Problem. Travers cites Japanese results for this problem that indicate that "the first problem is most likely to reflect students' initial impression of the given problem" (p. 6). Problems formulated by subjects were divided into four categories: problems similar to the given problem (a repeating pattern is given and a number of parts must be determined given the repetition of the pattern), basic arithmetical problems, problems involving simple counting or measuring, and all other problems. No examples of problems in each category are given.

The analyses show that sixth grade subjects (56%) more than fourth grade subjects (25%) were able to formulate problems similar to the Matchsticks Problem, while 58% of fifth grade subjects produced similar problems. For fourth grade subjects, 50% formulated problems in the "other" category, many of which were unintelligible, though subjects may have intended to create problems which would have been classified in other categories. For this reason, Travers did not further compare, by grade level, problems made by subjects in the other categories.

Boys (65%) more than girls (50%) created problems in the "other" category among fourth-grade subjects. However, at the sixth grade level, boys produced more problems similar to the given problem (62%) than girls (52%). Girls formulated more arithmetical problems (17%) than boys (9%). For both fourth-grade boys and girls, those who got a correct solution produced similar problems (35%) more frequently than subjects who answered incorrectly (22%). Apparently these subjects saw a pattern or structure which facilitated formulation of a problem similar to the given one.

For sixth grade subjects, 58% and 54% of subjects who got a correct or incorrect answer, respectively, formulated similar problems. For those who answered incorrectly, gender made little difference; however, for subjects answering correctly, 68% of boys and 50% of girls formulated similar problems.

Travers further analyzed data for subjects who created problems categorized as similar to the Matchsticks Problem. The vast majority of such subjects posed simple extensions of the original problem (e.g., how many matchsticks if the number of squares is, say, 12). The number of subjects who posed problems asking something different was very small - too small to analyze in terms of gender, grade, or rate of correctness. The most common of such posed problems was the converse problem (i.e., given, say, 52 matchsticks, how many squares can be made?).

The second dimension of analysis concerned the shared-side characteristic of the Matchsticks Problem. Subject-formulated problems were categorized as Retaining Shared Side, Changing the Shared Side Characteristic, Eliminating the Shared Side, and Unclear on the Condition of Shared Side. For both fourth and sixth grade subjects: 33% and 34%, respectively, retained the Shared Side Characteristic, 10% at each grade level changed it, and 48% of fourth and 44% of sixth grade subjects did not include the condition. There were similar results for grade five subjects. No gender differences were found. Correctness, however, was found to be related to the use of the shared-side characteristic: 37% of subjects getting a correct answer used the shared-side characteristic in their problems, while 25% who got incorrect answers retained the condition. The difference was consistent for both gender and grade level. Travers comments that since one key to getting a correct answer involves the shared side idea, it is expected that subjects who solved the problem correctly would more likely include the condition in formulating their problems (p. 10).

Some subjects also formulated problems with two or three dimensional arrays of squares (i.e., two or three rows stacked up and sharing horizontal as well as vertical sides) or "special forms" in which unit figures form a pyramid, a circle, or a set of concentric circles. However, the number of such problems was small and "one-row" problems dominated across genders, grade levels, and rate of correct responses given by subjects. In formulating problems, subjects nearly always used a figure or illustration.

In analyzing the results for the problem chosen as the best one, Travers organized data into five categories: hardest, easiest, content of the problem (i.e., because it has fractions), value of the problem (i.e., different or has educational value), and all other responses (i.e., "fun", "neat" or "best"). The analyses showed clearly that for both genders and grade levels and for rate of correctness, the reason subjects most frequently gave was that it was the hardest.

Finally, tabulation of questionnaire responses showed that:

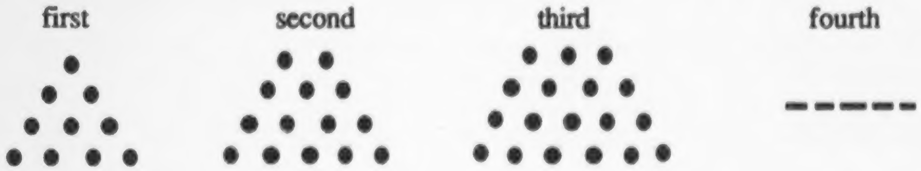
- Subjects found the problem interesting (slightly more than 50%) at both grade levels.
- Subjects found the problem easy at both grade levels (50% for fourth grade and 55% for sixth).
- Fourth-grade (52%) and sixth-grade (34%) found the problem different from problems in their textbooks, and few subjects thought the problem was the same.
- 62% of fourth grade and 50% of sixth grade subjects "like Math". Few subjects at either grade level said they "dislike Math".
- 49% of fourth-grade and 31% of sixth-grade subjects feel they are "good at Math." Few subjects at either grade level feel they are not good at Math. For subjects at both grade levels who got the correct answer to the Matchsticks Problem, 49% reported they were "good at Math" while 37% of those with incorrect answers reported the same. The difference was more marked for boys than girls at each grade level.
- Slightly more than 50% of students liked the problem more than textbook problems.
- 49% of fourth and 75% of sixth-grade subjects responded that they had seen problems like this one before, which seems inconsistent with performance.

Marble Pattern Problem / Grades 6,8,11

Data for the Marble Pattern Problem were analyzed and reported by Kantowski (1991). The problem was presented to subjects as depicted below, followed by five solution spaces. Ways of students' thinking about the problem are given in Appendix C.

Problem I

Marbles are arranged as follows:



Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) How many marbles are there in the fourth place?

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN. Write your way of solution and the answer.

(2) How many marbles are there in the sixteenth place?

Show your way of solution and your answer.

(3) Try to find a formula for finding the number of marbles in the one hundredth place.

There was a total of 794 students in the U.S. sample: 179 at grade 6, 368 at grade 8, and 247 at grade 11. No numbers for gender are given. In reporting her analyses, Kantowski states and provides results for the ten questions that follow.

1. What is the total number of approaches used by students at each grade level to find the correct response to question 1 in the Marble Pattern Problem?

   For this statistic, students were given one point for each approach they wrote. If the same solution process was used more than once, but in a slightly different format, the student was given more than one point for one process. The maximum number of approaches for any given student was 6 and the minimum was 0. The only significant difference in total number of approaches was by grade-level - sixth grade students did not show as many approaches as students at grades eight and eleven.

Total number of approaches by grade level for U.S. students - Question 1 for Marble Pattern Problem

Grade	Mean	Standard Deviation
6	1.92	1.46
8	2.20	1.21
11	2.09	1.11

2. How many different approaches did students in each grade level use to find the correct response to question 1 of the Marble Pattern Problem?

For this statistic, students were given one point for each different solution process used. The maximum number of different approaches for any student was 5 and the minimum was 0.

Again, the only significant difference in total number of different approaches used was by grade level - sixth grade students had significantly fewer different approaches than students at grade levels eight and eleven.

Total number of different approaches by grade level for U.S. students - Question 1 for Marble Pattern Problem

Grade	Mean	Standard Deviation
6	1.58	1.15
8	1.91	1.04
11	1.91	1.00

3. What percentage of students at each grade level obtained at least one correct solution for question 1, question 2 and question 3 of the Marble Pattern Problem?

Since question 3 on the 6th and 8th grade survey was different from question 3 on the 11th grade survey, there can be no valid comparisons between grades 6 and 11 and grades 8 and 11 on question 3. The lower two grades were asked to find the number of marbles in the 100th place, whereas the 11th grade students were asked to find a formula for the number of marbles in the n-th place.

82% of all 6th grade students were correct on question 1
93% of all 8th grade students were correct on question 1
96% of all 11th grade students were correct on question 1

26% of all 6th grade students were correct on question 2
53% of all 8th grade students were correct on question 2
68% of all 11th grade students were correct on question 2

17% of all 6th grade students were correct on question 3
30% of all 8th grade students were correct on question 3
40% of all 11th grade students were correct on question 3

91% of all female students were correct on question 1
92% of all male students were correct on question 1

55% of all female students were correct on question 2
48% of all male students were correct on question 2

26% of all 6&8th grade female students were correct on question 3
25% of all 6&8th grade male students were correct on question 3

36% of all 11th grade female students were correct on question 3
44% of all 11th grade male students were correct on question 3

4. How often was each approach used at the three grade levels for question 1 of the Marble Pattern Problem? The results appear as percentages for each grade, for each approach.

   Approaches 2 (which was 4+5+6+7) and 3 (which was 7+6+5+4) were scored separately since many students seemed to focus on the bottom line of the trapezoid and therefore went to great lengths to determine the number of marbles on the fourth row, and then add upwards. Net gain indicates that the student visualized one row being taken from the top and a new one added to the bottom (–4+8=+4). The pattern approach (10, 14, 18, 22) was indicated by a table. The +1 each row approach was sometimes indicated with a picture and sometimes not. An attempt was made to distinguish between students who used a picture for this process and students who did not, but it was always possible since, in many cases, a picture might appear with another solution but not with the +1 process. Unique solutions were those used by 5 or fewer students - in the 6th and 8th grades, a formula was considered unique since few students used that approach.

Approaches used by students for question 1 for the Marble Pattern Problem

	Approach	Grade 6 (%)	Grade 8 (%)	Grade 11 (%)
1	counting	31	31	43
2	adding 4+5+6+7	20	23	27
3	adding 7+6+5+4	6	3	7
4	net gain	1	4	4
5	pattern +4	38	59	52
6	+1 each row	37	48	23
7	group 10+3(4)	11	11	9
8	group 4(4)+6	7	8	11
9	group other	3	2	2
13	unique	4	3	5
14	formula	-	-	8

Note: Percentages do not add to 100 because students used more than one approach.

5. How often was each approach used at each of the three grade levels for question 2 of the Marble Pattern Problem? The results appear as percentages for each grade, for each approach.

Approaches used by students for question 2 for the Marble Pattern Problem

	Approach	Grade 6 (%)	Grade 8 (%)	Grade 11 (%)
1	counting	2	4	1
2	add 16+17+18+19	12	14	17
3	add 19+18+17+16	1	1	5
4	pattern table	4	18	25
5	16(4)+6	2	5	8
6	15(4)+10	2	5	8
7	22+4(12)	0	0	0
8	Unique	0	0	0

6. How often was each approach used at the 6th and 8th grade levels for question 3 of the Marble Pattern Problem? The results appear as percentages for each grade, for each approach.

Approaches used by students for question 3 for the Marble Pattern Problem

	Approach	Grade 6 (%)	Grade 8 (%)
1	100+101+102+103	11	14
2	Pattern with gaps	0	2
3	4(100)+6	1	4
4	4(99)+10	3	5
5	22+4(96)	2	2
6	Formula	1	3

7. How often was each approach used in grade 11 for question 3 of the Marble Pattern Problem? The results appear as percentages for each grade, for each approach.

Approaches used by students for question 3 for the Marble Pattern Problem

	Approach	Grade 11 (%)
1	Pattern - Table (with gaps)	2
2	4n+6	18
3	4(n–1)+10	11
4	n+n+1+n+2+n+3	7
5	Unique	2

8. Of those students who found correct solutions for both question 1 and question 2, what percent used an approach used in question 1 to solve question 2?

   69% of all students having a correct answer for questions 1 and 2 used an approach used for question 1 to solve question 2.

9. Of those 6th and 8th grade students who found correct solutions for both question 2 and question 3, what percent used an approach used in question 2 to solve question 3?

   90% of 6th and 8th graders who were correct on questions 2 and 3 used the same approach on both problems.

10. What percent of students at each grade level used incomprehensible approaches for question 1?

20% of all 6th grade students used incomprehensible approaches for question 1.

30% of all 8th grade students used incomprehensible approaches for question 1.

13% of all 11th grade students used incomprehensible approaches for question 1.

22% of all females used incomprehensible approaches for question 1.

23% of all males used incomprehensible approaches for question 1.

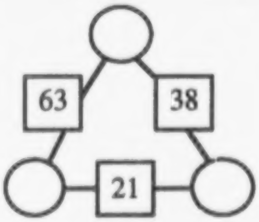
Arithmogons Problems / Grades 8 and 11

Data for the Arithmogons Problems were analyzed and reported by Becker and Owens (1991). The problems were presented to subjects as depicted below, followed by five solution spaces for the three-sided Arithmogon problem and space, as shown, for the four-sided Arithmogon Problem. Ways of students' thinking about the problem are given in Appendix D.

Problem II

Given a three-sided arithmogon as in the figure below. We put three numbers in the three ☐ - the number in each ☐ must equal the sum of the numbers in the two ○ on either side.

Find the numbers for ○ at each corner. The numbers in ○ may be negative numbers.

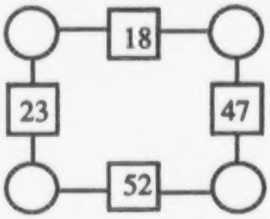


Do not erase anything you write down, just draw a line through anything you feel is in error.

FIND THE ANSWER IN AS MANY DIFFERENT WAYS AS YOU CAN.

(2) Now change to a square (four-sided) arithmogon as in the figure below. The number in each ☐ must equal the sum of the numbers in the two ○ on either side.

Try to find the numbers for ○ at each corner



There were 368 (178 male and 190 female) eighth-grade students in mathematics classes in the Carbondale (IL), Champaign/Urbana (IL), Pittsburgh (PA), Gainesville (FL), and Athens (GA) centers. There were 246 (124 male and 122 female) eleventh grade students from the same centers, except Champaign/Urbana (IL). U.S. results were compared to those of the Japanese, sample for which included 189 (96 males and 93 females) eighth-grade students and 234 (135 males and 99 females) eleventh-grade students.

A method of scoring devised by Japanese researchers (Senuma and Nohda, 1989) and reported in Miwa (1991) was used. At each grade level, problems were scored correct/incorrect/not-attempts, number of different ways of solving the problems, ways leading to correct/incorrect solutions, and success of students on both problems. Two persons independently scored student scripts. Tabulations were made of student responses on the questionnaire and were related to overall performance on the problems; similarly for the teacher questionnaire.

A brief, concise summary of U.S. and Japanese results is given below.

Teacher Questionnaire

U.S.:	Students accepted the problems, liked them, found them challenging, and represented their best effort. Teachers liked the problems, thought them to be "thinking" problems, said more such problems are needed in the curriculum, said their students gave their best effort, and that this is the first such experience of their students with such problems.
Japan:	No information

Student Performance on Problems

		8th Grade		11th Grade
Problem I:	U.S.:	15% got correct solution (more males correct than females)	U.S.:	46% got correct solution (male/female differences small)
	Japan:	39% got correct solution	Japan:	90% got correct solution
Problem II:	U.S.:	26% got one correct solution	U.S.:	55% got one correct solution
	Japan:	39% got one correct solution	Japan:	25% got one correct solution

Approaches Used

		8th Grade		11th Grade
Problem I:	U.S.:	Almost all subjects used Trial & Error	U.S.:	Almost all subjects used Trial & Error (only 6% used simultaneous equations and got a solution)

		8th Grade		11th Grade
		Number of subjects who used more than one approach negligible		Number of subjects who used more than one approach negligible
		Large number did not understand the problem Many did not know how to use negative numbers		Large number did not understand the problem Many did not know how to use negative numbers, though fewer than 8th grade subjects
	Japan:	19% used simultaneous equations 44% used Trial & Error 29% used other approaches 8% no solution Whatever approach used, was used successfully	Japan:	82% used simultaneous equations 8% used Trial & Error 7% used other approaches 3% no solution Whatever approach used, was used successfully
Problem II:	U.S.:	Almost all used Trial & Error Large number did not understand/ try problem Very few subjects noticed there was more than one solution	U.S.: < 10/614 > (2%)	Almost all used Trial & Error About half did not understand/ try problem Very few subjects noticed there was more than one solution
	Japan:	42% used Trial & Error 9% used simultaneous equations 10% used other approaches 40% got no solution	Japan:	12% used Trial & Error 64% used simultaneous equations 5% used other approaches 20% got no solution

Student Questionnaire

U.S.:	Like math, feel they are good at math, found the problems interesting, thought the problems difficult, felt problems different from textbook problems and like problems less than textbook problems	Like math, feel they are good at math, found the problems interesting, thought the problems difficult, felt the problems different from textbook problems, and like the problems more than textbook problems.

	8th Grade	11th Grade
Japan:	24% like, 27% dislike, and 49% neutral about math; feel not good at math, problems not interesting, problems difficult, problems different from textbook problems and like less than textbook problems	18% like, 29% dislike, and 53% neutral about math; not good at math, problems interesting and not interesting in equal percents and remainder neutral (39%), problems difficult, problems different from textbook problems, and small tendency towards liking problems more than textbook problems (half neutral)

For the first problem, at the eighth grade level, Japanese subjects performed much better than U.S. subjects, with no gender differences for the Japanese and small differences for the U.S. At the eleventh grade, the differences are even more striking, favoring the Japanese, with no gender differences for either sample. For the second problem, Japanese eighth grade subjects again perform better than U.S. with no gender differences. At the eleventh grade, the results are reversed, with no gender differences in either sample. It is not clear why the reverse results occurred, though a possible and likely explanation is that since there were 15 minutes allowed for subjects to do both problems and Japanese subjects commonly used simultaneous equations for the first problem (82%), perhaps there was too little time left for the second one (for which 64% used the same approach with four variables, in contrast to U.S. subjects who used Trial and Error commonly, which is an approach that would work starting with any integer). Further, Japanese eleventh grade students already have a good familiarity with algebraic methods which they begin to learn in the seventh and eighth grades. In contrast, U.S. students do not have this same degree of familiarity, usually beginning study of algebra in grade 9. It is also noteworthy that U.S. subjects, at both grade levels, displayed difficulties with negative integers, in contrast to the Japanese. Finally, we also note that (1) Japanese, more than U.S. subjects, were more successful whatever the approach used (accuracy is important!), (2) at both grade levels, U.S. subjects more than Japanese, fairly strongly like math and feel they are good at math, but the Japanese perform better; while, in contrast, Japanese subjects strongly feel either neutral about or dislike math, and (3) Japanese subjects appear to be more "fluent" in mathematics shown by a wider diversity of approaches used in solving the problems.

Area of Squares Problem / Grade 11

Data for the Area of Squares Problem were analyzed and reported by Wilson (1991). The problem was presented to subjects as depicted below. Ways of students' thinking about the problem are given in Appendix E.

Problem I

Pick a point P on the line segment AB and make squares: one side of one is AP and one side of the other is PB . Where should the point P be located to satisfy the condition that the sum of the areas of the two squares is a minimum?

Do not erase anything you write down, just draw a line through anything you feel is in error.

(1) Write a way of solution and the answer to the above problem.

(2) Now make up your own problems like the one above and write them down. Make as many problems as you can. You do not need to find the answers to your problem.

(3) Choose the problem you think is best from those you wrote down above and write the number of the problem in the space: __________

Write the reason or reasons you think it is best.

The problem was attempted by 247 U.S. algebra II students. No numbers for gender are given. The Japanese report, giving a preliminary analysis of data for this problem for Japanese 11th graders, showed that only 14% gave correct answers that included complete solutions (Wilson, p. 6). Wilson points out that, in the Japanese scoring scheme, inductive approaches such as calculating areas or developing intuition by drawing pictures were called "inappropriate reasoning, correct answer in the end" (p. 6).

Problem booklets for one algebra II class were examined and the following scoring categories were developed by Wilson. Independent scorers showed close agreement when using these categories for that class, so that 247 booklets were scored accordingly:

A. The student produced a drawing or sketch that was a reasonable interpretation of the problem.
B. The student produced some work in addition to or without a drawing to show some understanding of the problem.
C. The student produced an argument, line of reasoning, or sequence of steps leading to a correct answer.
D. The student produced the correct answer.
E. The student explicitly said "I do not understand [the problem]."

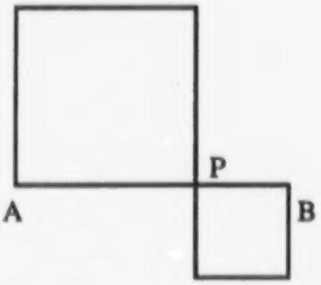
The percentage of students in each of the categories is shown in the table below. Data are given for each U.S. center as well as for the total. An additional line in the table presents data for 48 preservice secondary mathematics teachers.

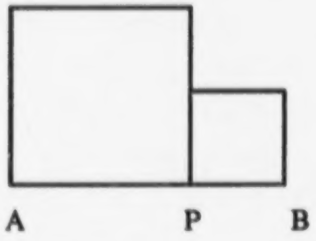
Types of Responses to the Areas of Squares Problem*

	N	A	B	C	D	E
Center 1	48	81.3	39.6	39.6	85.4	0.0
Center 2	33	69.7	51.5	39.4	81.8	3.0
Center 3	50	80.0	38.0	38.0	80.0	0.0
Center 4	116	85.3	65.5	46.6	73.3	7.8
TOTAL	247	81.4	53.0	42.5	78.1	4.0
Preservice Sec. Math Teachers	48	56.3	68.8	72.9	75.0	0.0
Japanese 11th Grade					84.0	

N = Number of Students Tested
A = Made a reasonably correct drawing or sketch
B = Something beyond the drawing to show some understanding of the problem
C = Reasoning or argument presented
D = Answer "In the middle," "at the midpoint," or equivalent
E = Explicitly write "I do not understand [the problem]"
* The Japanese did not use this scoring scheme. Data were not collected in one U.S. center.

Draw a figure. Wilson reports that more than 80% of students produced a drawing that was a reasonably correct interpretation of the problem statement. These drawings were about equally divided between drawing the squares on opposite sides of AB and drawing them on the same side, as below:





Many students made only a drawing with P at the midpoint and wrote a correct answer. It would seem that these students already had some intuition that P should be at the midpoint and made the drawing to fit that intuition. This is reasonable in that the symmetry of the situation would allow the midpoint as the only unique placement for P, but no student made an explicit argument of the symmetry.

It is encouraging that 80% of students could make a problem translation from the verbal mode to an iconic one, as reported by Wilson, but it is discouraging that so many students reasoned only from their drawing. For preservice teachers, only 56% made a drawing. In part this was probably due to a number of them producing calculus solutions -- setting up a function, taking the derivative, setting the derivative equal to 0, and solving -- without making a drawing.

Evidence of understanding: This category was checked if the student made a verbal restatement of the problem, drew multiple drawings to show P could vary along AB, or provided an incomplete argument before writing the correct answer. The category was used to categorize cases in which the student made some progress towards a solution beyond a single sketch of the situation.

Reasoning or argument presented. 14% of Japanese students presented a correct answer with mathematical reasoning (p. 8). The corresponding result for U.S. students would be 0%. Of the 247 students, none produced an algebraic equation to represent the problem. One student wrote $x^2 + y^2 \leq ((x+y) / 2)^2$ but did not connect it to anything in the drawing (which is just as well since it is not true).

For U.S. algebra II students, about 42% presented reasoning or an argument leading to a correct answer. In every case the reasoning was inductive. The most common approach was to particularize the length AB and to calculate a sequence of areas, usually accompanied by a sequence of drawings. For Japanese students using "inappropriate reasoning," 26% calculated areas and 21% reasoned by drawing pictures.

Correct answer. The problem asks students where to locate P to minimize the two areas. In retrospect, if we wanted explanation and justification, we should have asked for it. About 78% of algebra II students produced a correct answer (i.e., "at the midpoint of AB" or something equivalent). Many drew a single figure and wrote an answer. 84% of Japanese students had a correct answer, of which 70% used "inappropriate" reasoning.

"I do not understand." This category came about when developing the scoring scheme with the one algebra class. It occurred only 10 times out of 247, and 9 of these 10 were in one center. One student wrote a correct answer and the comment, "This problem is a perfect example of how I know the answer but have no idea how to go about getting it." One suspects the student is not unique in this respect among the 247 algebra II students in the study.

Results for parts 2 and 3 of the problem, which ask students to make up their own problems, select the one liked best and to give the reason(s) will be included in a draft of the final report.

Note: Reports of analyses of U.S. data for each of the five problems used in the U.S.-Japan research, reported herein, were formulated from the drafts of the reports by Becker, Silver, Kantowski, Travers, and Wilson.

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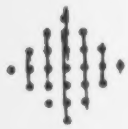
APPENDIX A

Marble Arrangement Problem

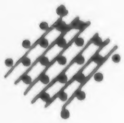
Some Possible Ways of Thinking About the Problem
[Nagasaki and Yoshikawa, 1989]

(a) Simple Counting

Vertically

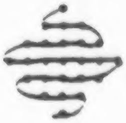


Diagonally

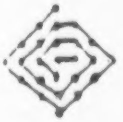


Drawing lines

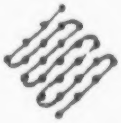
(1)



(2)

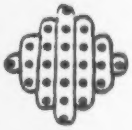


(3)



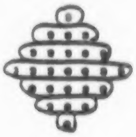
(b) Counting and adding

(1)

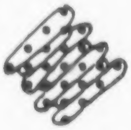


1+3+5+7+5+3+1 = 25
(1+3+5) x 2 +7 = 25

(2)



(3)

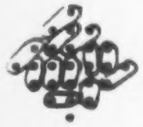


4 x 4 + 3 x 3 = 25

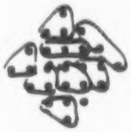
(c) Grouping

(1) Making groups of marbles of the same number and adding:

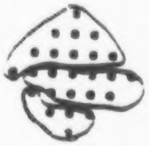
(1)



(2)

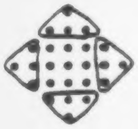


(3)



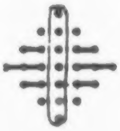
(2) Grouping into subgroups

(1)



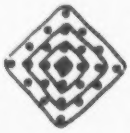
4 x 4 = 16, 3 x 3 = 9
16 + 9 = 25

(2)



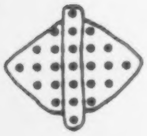
1 x 2 = 2
2 x 2 = 4
3 x 2 = 6
2 x 2 = 4
1 x 2 = 2
2 + 4 + 6 + 4 + 2 + 7 = 25

(3)



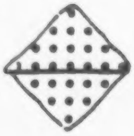
12 + 8 + 4 + 1 = 25

(4)



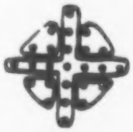
9 + 7 + 9 = 25

(5)



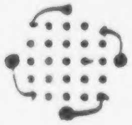
16 + 9 = 25

(6)



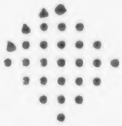
3 + 4 = 12
12 + 7 + 6 = 25

(d) Moving marbles



5 x 5 = 25

(e) Supplementing



7 x 4 = 28
28 − 3 = 25

(f) Explaining using algebraic expressions

1. 12 x 2 + 1 = 25
2. 4 x 4 = 16, 3 x 3 = 9, 16 + 9 = 25
3. 4 + 3 + 4 + 3 + 4 + 3 + 4 = 25
4. 7 + 7 + 7 + 4 = 25
5. 7 x 2 = 14, 14 + 11 = 25
6. 3 x 7 = 21, 21 + 4 = 25
7. 9 x 2 = 18, 18 + 7 = 25
8. 5 x 5 = 25
9. 4 x 7 = 28, 28 − 3 = 25
10. 16 x 2 − 7 = 25
11. 3 x 4 = 12, 12 + 7 + 6 = 25

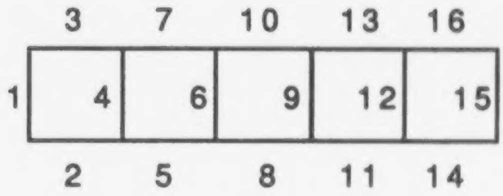
APPENDIX B

Matchsticks Problem

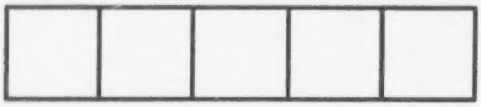
Some Possible Ways of Thinking About the Problem

[Hashimoto, 1986]

a)

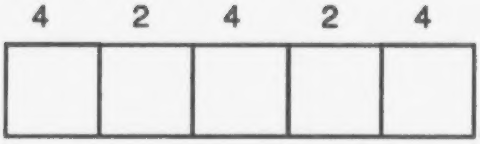

3
7
10
13
16
1
4
6
9
12
15
2
5
8
11
14

b)



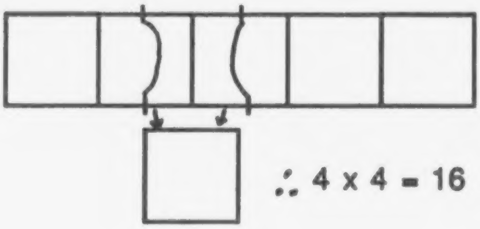
4 matches/square and 5 squares
5 x 4 = 20 WRONG!

c)



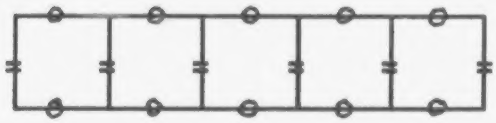
4 x 3 + 2 x 2 = 16

d)



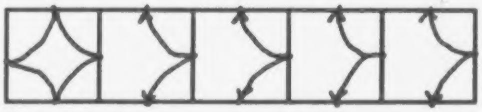
∴ 4 x 4 = 16

e)



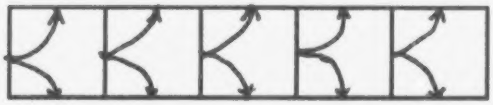
5 x 2 + 6 = 16

f)



4 + 3 x 4 = 16

g)



5 x 3 + 1 = 16

h) other ways?

APPENDIX C

Marble Pattern Problem

Some Possible Ways of Thinking About the Problem

a) Make a picture of the marbles in the fourth place and COUNT.

b) See a pattern, like:

1st	2nd	3rd	4th
10	14	18	22

c)
$$\begin{array}{r} 4 \\ 5 \\ 6 \\ \underline{+\ 7} \\ 22 \end{array} \quad \text{or} \quad \begin{array}{r} 7 \\ 6 \\ 5 \\ \underline{+\ 4} \\ 22 \end{array} \quad \text{or} \quad \begin{array}{r} n \\ n+1 \\ n+2 \\ \underline{+\ n+3} \\ 4n+6 \end{array}$$

d) 10, 10 + (1 x 4), 10 + (2 x 4), 10 + (3 x 4)

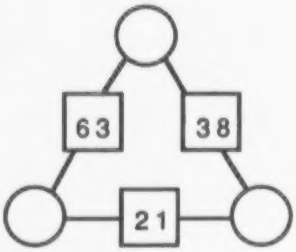
e) 6 + (1 x 4), 6 + (2 x 4), 6 + (3 x 4), 6 + (4 x 4)

APPENDIX D

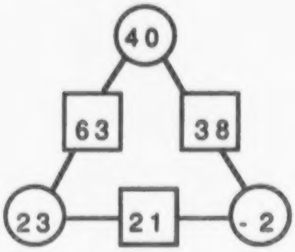
Arithmogons Problem

Some Possible Ways of Thinking About the Problem

Problem I:



Unique Solution



(1) Random Trial and Error

Here subjects might guess a number for the top ○ and, by subtraction and moving counterclockwise, see if they would end up with the same number in the top ○. Alternately, subjects might (a) work clockwise or (b) work both clockwise and counterclockwise starting with a guess in the top ○, to see if they end up in both directions with 21 at the bottom.

(2) Systematic Trial and Error

Here subjects might reason that the numbers in the top ○ and lower left ○ must add to 63. After picking a pair adding to 63, work around counterclockwise or clockwise, using subtraction, to see if they end up with the same number in the top ○. If not, pick a different pair and proceed similarly.

(3) One Equation in One Unknown

Let x represent the number in the top ○. Then the lower left ○ is 63 – x and the lower right ○ is 38 – x. The two must add to 21; so

$$(63 - x) + (38 - x) = 21$$

(4) System of Two Equations in Two Unknowns

Here subjects might let x represent the number in the top ○ and y the number in the lower right ○. Then $x + y = 38$ and $63 - x = 21 - y$; so

$$x + y = 38$$

$$x - y = 42$$

(5) Three Equations in Three Unknowns

Here subjects might let x represent the number in the top ○ , y the number in the lower right ○, and z the number in the lower left ○ ; so

$$x + y = 38$$

$$x + z = 63$$

$$y + z = 21$$

(6) By Adding 63, 38, 21 (Seeing a structure)

$$63 + 38 + 21 = 122$$

$$122 \div 2 = 61$$

$$61 - 63 = -2$$

or

$$61 - 38 = 23$$

or

$$61 - 21 = 40$$

(7) Difference of the two smallest □ 's (Seeing a structure)

a. Find the difference of the numbers in the two smallest □ 's.

b. Subtract the difference from the number in the largest □ .

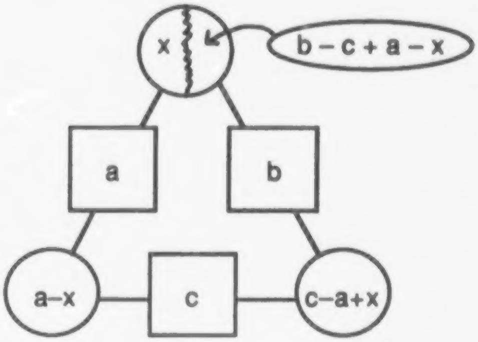
c. Divide the second difference by 2, which is one of the numbers in the ○ 's.

d. Add this number to the first difference to get the number for the next ○ .

e. Determine the number for the third ○.

(8) General Solution (Changing perspective and solving a "bigger" problem first)

Let **x** represent the number in the top ○ and let the numbers in the □ 's be represented by **a**, **b**, and **c**. Then, work counterclockwise.

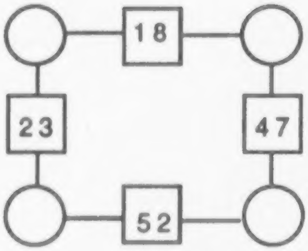


then, $x = b - c + a - x.$

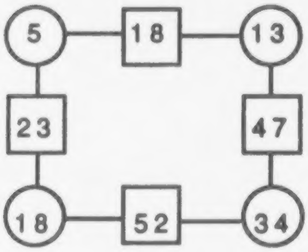
so, $x = \dfrac{a + b - c}{2} = \dfrac{63 + 38 - 21}{2} = 40.$

so, 40, 23, and −2 are the solution.

Problem II:



One non-unique Solution



It was anticipated that students would exhibit one or more of the following approaches to solving the problem.

(1) Trial and Error

Let the top left ○ be 5 (or any integer). Then the lower left ○ is 18; then the lower right ○ is 34; then the upper right ○ is 13; and 5 + 13 = 18.

Note: Will subjects recognize that starting with any number in any ○ will lead to a solution, and that there is more than one (infinitely) many solutions?

(2) Four Equations in Four Unknowns

Let **x**, **y**, **z**, **w** represent the numbers in the four ○ 's. Then

$$x + y = 23$$

$$y + z = 52$$

$$z + w = 47$$

$$x + w = 18$$

(3) Two Equations in Two Unknowns

Let **x** represent the number in the upper left ○ and **y** the number in the lower right ○ .

Then, $18 - x = 47 - y$

$23 - x = 52 - y$

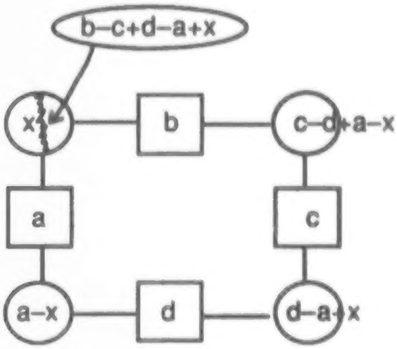
So, $x - y = -29$

$x - y = -29$

Therefore, there are infinitely many solutions.

(4) Addition of Pairs of Numbers in Opposite □ 's.

Will subjects see that 23 + 47 = 52 + 18 and, therefore, there are infinitely many solutions, or reason as follows?



So, $x = b - c + d - a + x$

So, $a + c = b + d$ (condition for a solution to exist)

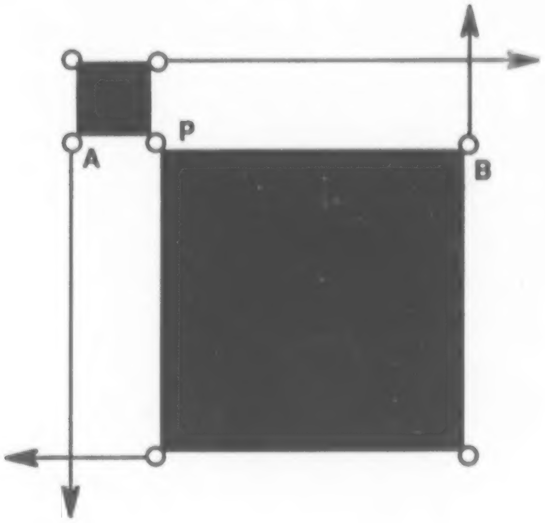
APPENDIX E

Area of Squares Problem

Some Possible Ways of Thinking About the Problem

[Wilson, 1991]

The problem asks for the minimum of the sum of two squares as shown below. If the sides of the squares are extended in our sketch to form a square of length AB on each side, four regions are formed: the squares AP^2 and PB^2, and the two rectangles each AP by PB. Now the total of the four regions is always AB^2. Therefore the minimum sum of the squares $AP^2 + PB^2$ occurs when the two rectangles have maximum area. But a rectangle has maximum area when it is a square or when $AP = PB$.

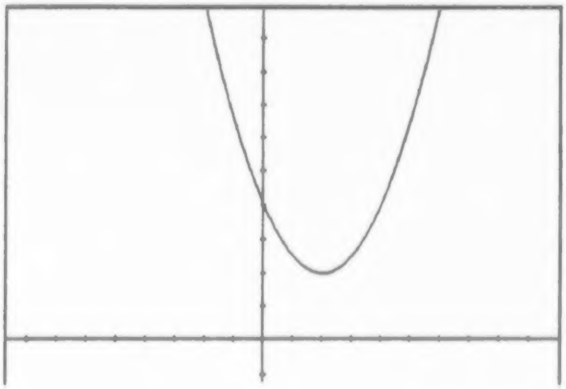


Variations on that approach include the following. Let $AB = x$ and $PB = y$. Then we want to minimize $x^2 + y^2$. By the Arithmetic Mean-Geometric Mean inequality,

$$x^2 + y^2 \geq 2xy \text{ , with equality iff } x = y \text{ .}$$

Therefore the sum of the two squares is always greater than the combined areas of the two rectangles except when $x = y$. So the minimum area occurs when P is the midpoint.

Another approach is to formulate the area as a function of a single variable. Let $AP = x$ and $PB = AB - x$. Then the area $f(x) = x^2 + (AB - x)^2$. This simplifies to $f(x) = 2x^2 - 2(AB)x + AB^2$. This might be recognized as a parabola with the following graph.



where the vertex is at $(AB/2, AB/2)$.

On the other hand $f(x) = -2x(AB - x) + AB^2$. By the Arithmetic Mean-Geometric Mean Inequality,

$$f(x) \le -2\left[\frac{x + AB - x}{2}\right]^2 + AB^2, \text{ with equality iff } x = AB - x$$

$$= -AB^2/2 + AB^2$$

$$= AB^2/2$$

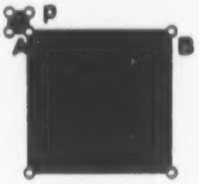
For the rare algebra II student that has had just enough cookbook calculus to take $f(x) = 2x^2 - 2(AB)x + AB^2$, find its derivative $f'(x) = 4x - 2(AB)$, and set equal to 0, the result is that he can conclude the function reaches a minimum when $x = AB/2$ without having to think about it.

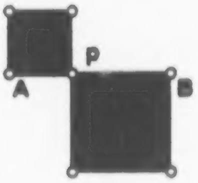
Another approach is to particularize the length AB and compute a sequence of values for $AP^2 + PB^2$ as P is placed along points on the line. Let AB = 10 and x = AP . Then the following table can be generated quickly.

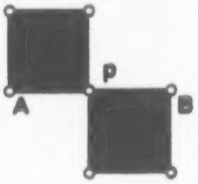
x	0	1	2	3	4	5	6	7	8	9	10
10 – x	10	9	8	7	6	5	4	3	2	1	0
sum	100	82	68	58	52	50	52	58	68	82	100

This provides good intuition that the desired location for P is at the midpoint of AB .

A variation is to draw a sequence of figures such as the following







Discussion of Professor Becker's Paper:

Sawada: Now we will begin the discussion of Professor Becker's paper. Are there questions or comments?

Hashimoto: I have a question for Professor Becker and Professor Fujii. What have you learned from this research, from the viewpoint of the counterpart country, and how are you going to implement it?

Becker: Should I respond first? Okay. As further background, to begin my response, when the U.S. group went to Japan before the data gathering began, the Japanese very graciously hosted us and made arrangements for our group of five researchers to make numerous visits to schools and mathematics classrooms and observe many problem solving lessons being taught by teachers. A common characteristic of nearly all lessons was the structure of the lesson. Of course, the problem had already been carefully researched; that is, it had been tried out in classrooms by teachers, so teachers had a good idea of how well the problem would work in terms of accomplishing the objective(s) the teacher had in mind. The lessons were structured as follows: the presentation of the problem took approximately 3-5 minutes, with a couple more minutes for making sure that students understood the problem and what they were expected to do; then in the next 20-25 minutes one of two things would happen: either the students worked individually on the problem in a whole-class setting, or individually for a few minutes and then formed small groups (and there are typically about 40 students in a Japanese classroom), or else they were divided up into small groups of 4 or 5 students to begin with to work on the solution to the problem; then the teacher had the students write their different solutions to the problems on the blackboard for everyone to see; then there was a comparing and discussing of the results; then there was a teacher summary of the lesson, and following that there might be a request by the teacher for students to write down what they learned in the lesson. Now, what does that have to do with Professor Hashimoto's question? At Southern Illinois University, in our NSF Teacher Enhancement project with middle school teachers, starting in the summer of 1990, we are implementing that structure of problem solving lessons. The vast majority of our teachers thought they were not capable of teaching that way. Classroom management is exceedingly important and, of course, the teacher has to understand the problem, the different approaches that the students may come up with, and an ability to discuss the mathematical quality of students' responses. But the teachers learn that after two or three lessons using the problem, they know what to anticipate to a very large extent and understand the mathematical significance in students' responses. The thirty teachers who are with us for four weeks of intensive work then implement this problem solving work, with some other work in the project, in their classrooms. They had to develop a plan for implementing this according to the reality of their classroom and their school. To a very large extent the teachers used the problems that we used in the summer institute and they used them according to the way we modelled teaching both with the teachers and grades 6-8 students in the summer. They learned that they could do it and they think that it's a very good approach. Another major activity in the summer was asking each teacher to write a lesson plan; that is, to select a problem and to develop a detailed lesson plan. Teachers resisted this. Many commented they had never written a lesson plan and asked why now? But all of them did it and, by and large, they developed very good lesson plans and at the end we had thirty lesson plans which we duplicated and all the teachers had a set of thirty problems with complete lesson plans written up that they could implement in their classrooms. All these lessons began with an "open-ended" problem, following the research done by Japanese Mathematics educators which was reported in our earlier seminar. That's one of the ways in which I have been influenced by the research. What I have described was part of the SIUC project proposal that went to the National Science Foundation and was reviewed very positively by the reviewers, with pretty supportive and constructive reactions. There are other things I could say, maybe just two other brief comments. One, we were very honored and very fortunate to have Professor Hashimoto work with us in our institute at Southern Illinois University earlier this summer. He taught the open-ended problem solving seminar to the first-year group. This was their second summer on campus. When he arrived here for the Seminar, he handed me the evaluations of the students in his seminar which were uniformly positive. Of course, they liked him as a teacher, and they also commented on something that I think is very important, namely, that he treated the problems mathematically in some depth. And the teachers were not used to that. I think maybe it's another aspect of the lesson to be learned from this research. Then, one other comment: We have a fairly extensive evaluation of this program at our university. We are looking at pre and post changes in attitudes towards mathematics, problem solving and use of technology, teachers' beliefs about mathematics, cognitive problem solving abilities, and assessing changes that are actually taking place in the classroom, using a comparison group. In general, from pre to post in the summer institute, there were significantly improved attitudes towards mathematics and problem solving, significantly improved performance with respect to knowledge, skills, and ability to solve problems, and significant differences from pre to post on teachers' beliefs about mathematics. The teachers' beliefs measure that we are using was a measure developed in this U.S.-Japan collaboration. Jim Wilson was in Japan about a year and a half ago and he worked with Professors Miwa, Fujii, Sugiyama and Sawada and others in developing this measure. We've used it now in the second year with very high measures of reliability for the different scales.

Of course, there were other things learned from the different components of the research. For example, we learned that, at the various grade levels, Japanese students have more technical knowledge of mathematics, a greater "fluency" in solving problems (i.e., can think about the problems in several or many ways, in contrast to U.S. students), express their ideas in mathematical notation more easily, and use trial and error less frequently than their U.S. counterparts. There are other interesting findings which should be further studied, for example, with respect to "liking math," "finding it easy," and so forth which are included in Professors Miwa and Fujii's paper. But let me stop here.

Sawada: Thank you. Any other comments?

Miwa: I would like to give a personal opinion about some of the findings. Professor Becker has given his personal opinion about what he thought of (1) Japanese students doing very well, in general, and that he believes this is a result of the guidance of the Japanese education in mathematics. However, I want to comment on another aspect. This has to do with (2) the aspect of students making up their own problems. Professor Fujii didn't see much creativity among Japanese students, whereas many of the problems made by the American students were ones he never expected, and so he feels that the American students had more freedom or creativity as far as math is concerned. Further, (3) attitudes toward mathematics. While it has been mentioned many times that the Japanese students do well in math, most of them say they don't like mathematics and don't think they're doing well, whereas the American students even though they do poorly, they still like math. This is also a point that Japanese education has to do something about.

Sawada: I think that is a very good point. Another question?

Miwa: Professor Fujii has a further comment.

Fujii: Here are some statistics about Japanese student performance. The first graph shows that, in the fourth grade, many students say they like math, but by the eleventh grade it goes down; whereas, in the fourth grade not very many students dislike math, but as the grade goes up they tend to dislike math more. The second diagram shows information about student thinking about doing well in math or not and their confidence. As the grade level goes up, it seems like they lose confidence and this is the biggest problem Japanese students have. This is something we have to deal with. They are doing well now, but it is not clear that they will keep doing well in the future. This is the big problem in Japanese mathematics education.

Sawada: Any other comment?

Fey: Is the situation any different from other subjects in the Japanese curriculum; is this a pattern that is true for Japanese language, social studies, science, and so forth? Is it different for mathematics from other subjects?

Sawada: Well, this tendency of more dislike is also true for Japanese language, math (of course), science, and social studies as well. The reason is that as the students go up in the grades, the content of the subject area get harder and so students think that they cannot really do well.

Uetake: Well, the new curriculum has been recently released. It addresses the value and beauty of mathematics and that this should be recognized and teachers have to work on how the student can learn to appreciate that.

Dugdale: On the first marble problem (fourth grade), the predominance of verbal expressions among the U.S. students and mathematical expressions used by Japanese students has been mentioned. I am puzzled by that. I wonder what speculation has been made about what in the curriculum accounts for this difference.

Becker: The reaction I have is that I simply have not found many U.S. teachers that emphasize to students the importance of representing their thinking mathematically when solving a problem; so, for example, in the marble arrangement problem, students commonly simply write a sentence or phrase or make a drawing that represents how they thought about the problem. Teachers just didn't seem to know the importance of expressing thinking in mathematical notation, and emphasizing this. I think it's maybe a little bit more common, but not common enough, to have students actually write down how they solved the problem or represent in writing their thinking about a problem. And that's something that the teachers I work with at all levels feel that they should spend more time on and that, in fact, it's important that kids are able to qualitatively examine and evaluate their own thinking and to express it using mathematical notation or expression.

Sakitani: For problems that require a solution using mathematical expressions not verbal; in the U.S., do you ever ask such questions, to answer using verbal or mathematical expressions?

Becker: I'm not sure how to characterize the role of verbalization. Maybe some members of the U.S. group might want to comment on that.

Demana: I'll take a crack at it. We've done a lot of work at Ohio State University (and Jim Fey at Maryland) with trying to get students to think about writing an expression or an (equation?), writing the math phrase given by keying a calculator and this is not an easy task and not at all reflected in the curriculum anywhere. Our curriculum is very answer-oriented and all the multiple-choice tests emphasize that too. We're very dominated by it. We also try to get kids to verbalize in an attempt to write math expressions and that is another hard thing because of this preponderance of answering by picking one of the multiple-choice responses.

Fujii: In Japanese classes as a daily and routine activity, after students give a correct answer, they then compare the ways of solving the problem and use mathematical expressions. Therefore, this is just a routine matter to most Japanese students. As a result, it can confuse the research results.

Nohda: Through this U.S.-Japan collaborative research, the Japanese have found out more about math education in the U.S. about the relationship between students and teachers. This has great merit for us. As John Dewey says, in education experience accumulates and, for the development and improvement of mathematics teaching, this is what is important in our collaboration. Certainly Japanese math teachers study their teaching materials very well and then they teach math to the students very effectively;

but, for the emotional side, whether students like the subject or not, this is neglected. For the Japanese side, we need to learn what is real education, not just teaching math; whereas from American side you can probably learn more about how to teach math. In this way, we can both benefit from this collaborative research work.

Sawada: Thank you, Professor Nohda, for your comment. And thanks to all of you for your many questions. But, I'm sorry our time is gone, and I have to stop now.

End of Discussion

AN OVERVIEW OF COMPUTER USE FOR MATHEMATICAL PROBLEM SOLVING IN JAPANESE SCHOOLS

Tatsuro Miwa
Institute for Applied Optics
Tokyo, Japan

1. Introduction

In March, 1989, the Japanese Ministry of Education released the revised Course of Study, which emphasized to foster students to cope with big changes assumed in the 21st century, and it will be put into force in 1992 for elementary, in 1993 for lower secondary and in 1994 for upper secondary schools, respectively. Among features in the new Course of Study, introduction of computers into school education is distinguished. Recently the Ministry of Education and local governments are eager in equipping computers in schools in order to prepare for implementing the new Course of Study, and computer use is expected to become very popular in Japanese schools in a few years. Today we are in the midst of transition.

In this paper, the author would like to review briefly recent trends and the present state of computer use in Japanese school education in general, and in mathematics education in particular, and examine computer use for mathematical problem solving. The paper consists of three sections except for the Introduction: section 2 will present recent trends in computer use in school education in Japan and it will give general background for Japanese situation. In section 3, the focus is mainly on computer use in mathematics education and we will present computer use in mathematics in general and those in the revised Course of Study in particular. The Course of Study is a standard for the national curriculum set by the Ministry of Education. Then we will examine computer use in teaching and learning mathematics and present examples in lower secondary school mathematics. In section 4, we will concentrate on computer use for mathematical problem solving, especially on its process and its teaching in Japanese classrooms.

This paper will be focused on general education, but not on vocational education.

2. Recent Trends of Computer Use in School Education

In Japan, computer use in education, except for a few pioneers, was not popular in almost all schools until the first half of the 1980s, despite the fact that our country was famous as a 'high technology' country, especially in the field of electronics.

In the middle of the 1980s new tendencies emerged. We can see them in the reports of the

National Council on Educational Reform, Government of Japan, and those of the Committee of Experts on School Education to Cope with the Information Society, the Ministry of Education. Based upon these reports and others, the Curriculum Council of the Ministry of Education had made a final report for the revision of the Course of Study. In these reports, school education to cope with the new information age, in which the computer is assumed to play central role, is emphasized.

In this section we will present an outline of the reports of the above councils and committees and computer use given in the revised Course of Study. Finally we will look at preparation for implementation of the new Course of Study in schools.

(1) School education to cope with new information age

The National Council on Educational Reform mentioned the following three principles for education to cope with the information age in the second report which was released in August, 1986 (The National Council on Educational Reform, 1986):

a) We should work seriously to develop education compatible with the dissemination of information media in society.
b) The potential of information media should be utilized to invigorate and stimulate educational institutions.
c) The potential danger of the dark side of the spread of information media should be compensated for with education, while the bright side should be utilized for harmonizing the educational environment.

In addition, the Council mentioned in the report, "We should promote the utilization of information media in learning activities in elementary and secondary education and thus foster learners' ability to use information, i.e., 'information literacy'." , and emphasized that schools and various other educational institutions should deal with the development of information literacy in accordance with each learner's stage of development. By information literacy the Council means basic competency of individuals to select and use information and information media independently. These principles and emphasis of information literacy is surely fundamental in the education for the new information age.

Prior to the above report, the Committee of Experts on School Education to Cope with Information Society, the Ministry of Education, explained the following on computer use in school education in its summary report of discussion released in August, 1985 (The Committee of Experts on School Education to Cope with Information Society, 1985):

a) Computers should be introduced primarily so as to contribute to attainment of essential aims and objectives of school education. Originally school education is expected to help students acquire the fundamentals and to develop their intelligence, virtue and body harmoniously through human interaction between teachers and students and between students, and learning by experience and through real things are stressed. Use of computers should not be contradictory to the aims of school education but help to realize them, that is, use of computers should help students to understand fundamental concepts and to foster thinking ability and creativity as well as assist teachers' teaching function.

b) To foster the aptitude needed in a new information age should be emphasized. In a new information age, the new aptitude, which is not necessarily the same as in these days, will be needed, e.g., to understand computers, to use them correctly, and to express one's thought in a form different from the traditional one. In school education, the fundamentals for fostering this aptitude is required and basic faculty leading to future application and creation must be stressed.

c) Introduction of computers should be done in accordance with development of students. Therefore, objectives and methods will vary from elementary school to upper secondary school or university, e.g., in elementary school, stressed is to get students to become familiar with computers, and in lower secondary school, stressed is to use computers in teaching and learning of academic subjects as well as to teach basics of computers, allowing options in accordance with students' ability and interests.

d) To activate school education by introducing computers should be implemented and for this it is important to arrange the surrounding environment including development of excellent software and teacher education.

The Curriculum Council of the Ministry of Education released its final report in December, 1987. In the report, as one of the basic philosophies for the revision of the Course of Study the Council suggested, "To emphasize fostering students' will to learn spontaneously and ability to cope with the change of society actively" and stressed logical thinking ability, imagination and intuition as well as information literacy; i.e., competency to understand, to select, to process, and to create information and ability and attitude to use information media such as computers. In addition, the Council mentioned that the influences brought by the new information age should be taken into consideration in the Course of Study. (The Curriculum Council, 1987)

(2) Computer use in the revised Course of Study

In the revised Course of Study, based upon the final report of Curriculum Council, the term 'information literacy' is not used explicitly but what is assumed in it are to be dealt with in various school subjects.

To foster competency to judge, select, organize and process information and ability to create and communicate information is emphasized in Japanese language, social studies, mathematics and science, to be aware of the influence of computers in society is to be dealt with in social studies, health and physical education, industrial art and homemaking and moral education, to understand the basics of computers and to operate computers in mathematics, industrial arts and homemaking and to utilize information media, such as computers, skillfully in teaching and learning in all subjects, especially in mathematics, science, industrial arts and homemaking and fine arts. In particular, in lower secondary school the subjects of industrial arts and homemaking, a topic 'basics for information' is to be dealt with, and its contents are to understand basic construction and functions of computers, to do fundamental operations on computers, to make up simple programs, and to make use of information with appropriate software.

(3) Preparation for computer use in schools

In order to put the revised Course of Study into practice in schools all over the country, it is needed that all schools have facilities of computers, including suitable software and preparation of teachers to be capable of teaching and operating computers. The following are statistics by the Ministry of Education in the end of March, 1989:

Table 1 The Number of schools equipped with computers

	number of schools (A)	number of schools having computers (B)	Percentage (B/A)	Mean number of computers in a school	Mean number of software in a school
Elementary	24,658	5,172	21.0%	3.0	19.6
Lower Sec.	10,585	4,740	44.8	4.3	27.5
Upper Sec.	4,189	4,035	96.3	25.5	78.5

Table 2 The number of teachers who can operate or teach computers

	# of teachers (A)	# of teachers able to operate(B)	# of teachers able to teach(C)	Percentage	
				(B/A)	(C/A)
Elementary	426,418	32,612	6,496	7.6%	1.5%
Lower Sec.	268,361	38,898	10,051	14.5	3.7
Upper Sec.	204,661	61,774	27,342	30.2	13.4

(Note. The above statistics include data of vocational courses in upper secondary schools, where computer use was popular from the beginning of the 1980s.)

Viewing these tables, equipment of computer in elementary and lower secondary schools is very limited and insufficient, especially in elementary schools, and teachers able to teach computers are rare. The latter is extremely serious. From 1989, much effort has been done but more effort not only of financial aspects but of teacher education is needed in order to improve the situation in our country. For instance, concerning teacher education, various short courses have been and are undertaken by the Ministry of Education, local boards of education in prefectures and cities, academic societies, e.g. Japan Society of Mathematical Education and other institutions and companies. In addition, in pre-service teacher education new requirement was established, which includes the training in computer use.

3. Computer Use in Mathematics Education

We will focus on computer use in school mathematics education. No doubt, the computer has peerless power in mathematics education. First we will cast a glance on computers in mathematics education. It will play a role of introduction in this section. Then we will look at computer use in the revised Course of Study. These are not in act today but it will be implemented in the near future. Next, we will focus on computer use in school mathematics classes, and present examples in the lower secondary school in order to be more in details and concrete.

(1) Computer in mathematics education

When we look at computer, we are surprised at its peerless power. In fact, the computer has capability of fast and precise numerical computation, fast and precise symbolic-expression manipulation and fast and precise processing and analysis of a lot of statistical data and fast and precise drawing of function graphs and geometric figures of two and three dimensions. Someone might assume the computer to be a panacea for resolution of difficult-ties in mathematics education. Needless to say, it is not true. Pointed out is that no advantage is brought about automatically only by existence of the computer and that any difficulty on teaching mathematics cannot be solved by only setting computers in the classroom. That is, it is man but not the computer that resolves difficulties. The computer is not the subject but a powerful means. (Howson et al. 1986)

In mathematics, the computer is very powerful at inquiry and discovery; for instance, use of computer graphics, which enables visualization of various phenomena and of plane and spatial figures and their motion, displacement and transformation, facilitates to make conjectures intuitively. Further, the first step of the inductive paradigm "computation-- conjecture-- proof" will be applicable in possible cases, not only in geometry but also in arithmetic, algebra and calculus. This illustrates that in mathematics classes, the computer promotes students' mathematical activities to inquire into and discover mathematics, which is very important in mathematics education. Thus, computer use allows students to be active at given mathematical phenomena and gets them to study various topics and concepts and to behave making better use of mathematical ideas autonomously and independently. Depicting a phenomenon which is difficult to represent computer use enables students to widen their field of vision and enrich their understanding, and motivates students to practice discovery process.

Briefly speaking, computer use allows experimental aspect of mathematics in class. Mathematics is said to be different from natural science, but in inquiry and discovery facets mathematics and natural science have commonality, and experiment is an essential element. However, mathematics has a facet of demonstration or proof. It is crucial to balance experimental mathematics and formal mathematics. (Howson et al. 1986)

(2) Computer use in mathematics of the revised Course of Study

We will return to computer use in mathematics of the revised Course of Study which was released in March, 1989. (Ministry of Education, 1989)

In elementary school, no content related to computers is described explicitly, but in 'the construction of teaching plan and remarks concerning content' suggested is "At the fifth grade or later, the teacher should help children to adequately use the 'soroban' (Japanese abacus) or hand-held calculators, for the purpose of lightening their burden to compute and of improving the effectiveness of teaching."

In lower secondary school, representing procedures of computation, etc. using flow chart, binary system and expression of number in the form of ax10n are content in the eighth grade, these are related to the basics of computers. In 'the construction of teaching plan and remarks concerning content', suggested are "In the teaching of each domain, computers should be efficiently utilized as an occasion demands. This matter needs to be considered in the instruction by the experiment and observation, etc. ", and "In the teaching of numerical computation, the teacher should give consideration to improve the effectiveness of learning by having students use the 'soroban', or hand-held calculators etc. as an occasion demands."

In upper secondary school, mathematics is composed of six subjects, Mathematics I, Mathematics II, Mathematics III, Mathematics A, Mathematics B and Mathematics C. Among them, Mathematics I is required for all students but the others are optional. In Mathematics A, one of its topics is 'computation and computer' and operation of computer, flow chart and programming and calculation using computers are to be dealt with. In Mathematics B, one of its topics is 'algorithm and computer' and function of computer, program of various algorithms, e.g. Euclidian algorithm and calculation of root by iteration are to be dealt with. In Mathematics C, through using computers from the viewpoint of applied mathematical science, matrix and linear computation, various curves, numerical computation or statistics are to be dealt with. In 'the construction of teaching plan and remarks concerning content' of all six subjects, suggested are "The teacher should make active use of educational media such as computers, so as to improve the effectiveness of learning.", and "In the teaching of computation, the teacher should have students use hand-held calculators and computers as an occasion demands, so as to improve the effectiveness of learning." More details are given in Fujita et al., 1990.

We see that in the elementary school, computer use is not given explicitly but the hand-held calculator is to be used for improving effectiveness of teaching and learning, and in lower secondary school a few topics related to computer are given and computer use aims mainly at improving effectiveness of teaching and learning. In upper secondary school, we see that computer use in mathematics is done aiming at to understand computer and to operate it in simple cases in optional subjects as well as to improve effectiveness of teaching and learning in all mathematics subjects.

(3) Computer use in teaching and learning of mathematics

Concerning computer use in mathematics education, pointed out is that there are the following three forms (The Committee of Experts on School Education to Cope with Information Society, 1985):

A. Using computer for improving effectiveness of teaching and learning.
B. Teaching of computer literacy.
C. Using computer for making up instructional plans and materials.

Among these three we will focus on the A in the following, as it is explicitly suggested in the revised Course of Study.

Rigidly speaking, computer use in teaching and learning be divided into following two categories:

(i) computer as a means by which teachers teach mathematics effectively.
(ii) computer as a means by which students study mathematics spontaneously. Actually, in mathematics classes, however, these two are inseparable as a whole and we do not distinguish between them in the following.

Computer use in teaching and learning should be devised so as to foster students' thinking ability and to get students to positively act on computers with a critical mind as well as to let them answer the questions posed by computers. To use computers in teaching and learning of mathematics aims mainly to help students do mathematics more and better. Computer use facilitates to enrich mathematical experiences, which is crucial in introduction of abstract mathematical concepts, and provides opportunities in exploratory and discovery thinking. (Fujita et al., 1990)

It is important to utilize functions of the computer, such as simulation, graphics, information retrieval and processing in instruction, and to connect computer use to real experiences of students, such as observation, experiment and practices. Lessons involving these activities are expected to have flexibility in implementation of classroom practice and to give much more motivation, to deepen students' interest into lessons, to get students to enjoy thinking and to foster logical thinking ability, problem solving ability and information processing ability. (Ministry of Education 1990)

Computer use allows teachers' teaching methods to be more varied and flexible and has potential to provide effective means for teaching topics in which many students have difficulties with traditional teaching methods. Moreover, it is expected to facilitate teaching and learning suited to individual aptitudes of students, so as to enhance individual student's interest at learning, to foster individual student's ability, to get the student to establish his/her learning style and form better attitude and to consolidate student's fundamental knowledge and skills. (Ministry of Education 1990)

The following are examples of computer use in teaching and learning in mathematics classes:

(a) It presents a dynamic view in various areas of mathematics. For instance, in geometry, to represent motion and displacement of figures in the plane and in space and their transformation and configuration vividly is very effective for students to deepen their understanding and enhance geometric intuition.

(b) It enables to compute complicated computation and let students appreciate the utility of mathematics. Computer use makes possible to stress approach and results rather than process of computation in order to make conjectures and to verify them smoothly. Further, with computers, students can deal with easily problems from real situations which includes often dirty numbers. It leads students to realize a real utility of mathematics. The hand-held calculator is also effective in this respect and more available in schools.

(c) It assists students to do problem solving and help them think spontaneously. Computer assists students by representing on the display overtly and concretely what they have drawn mentally and

verifying what they have conjectured, and promotes their thinking to develop easily and smoothly.
(d) Using the simulation function of the computer, it is possible to repeat many times experiments in a short time and without any danger.

While computer use has much expectation, the following attention should be paid: If computer use extends to unnecessary and inadequate ranges, students would think that computer can do everything and would reduce the positive attitude of using their own hands and bodies and seeing nature and society through their own eyes. It leads students to avoid coming in direct contact with nature, man and society and to reduce a student's intellectual creativity.

For computer use in teaching and learning to be effective, the following remarks are important:

a) As well as taking the nature of mathematics as a school subject into consideration across school levels, it is needed to be in accordance with students' developmental stages.
b) While the function of the computer should be fully utilized, the computer should be coordinated with other educational media together in a suitable way.
c) Computer use in education should not replace education by real materials and education through real experiences, but supplement and reinforce them.
d) To use the computer appropriately in the classroom, it is needed to devise suitable arrangement of hardware and software and instructional form. In addition, it should be considered not to increase students' burden.
e) Instruction in class before and after computer use is important. Use of computer without appropriate instruction is meaningless. For instance, in observation on a computer display needed is an instruction in advance to make clear what to observe and what to do after observation.
f) It is desirable that students record observation of computer display including pictures, numerical values and expressions, and discuss based on the record of these observations. More desirable is for students to be habitual in these actions, because to record makes their observation more precise and has potential to generate new vision and way of thinking which computer use causes.

(4) Examples of computer use in lower secondary school mathematics

Here we will concentrate on computer use in lower secondary school mathematics and give several examples. (JSME 1990, Ministry of Education 1991).

(A) When students try to find out properties of geometric figures, i.e., to make conjectures and to prove them, it is very useful to see and grasp points, lines and circles etc. on geometric figures dynamically. Until now, teachers illustrated only typical scenes and left dynamics in geometric figures to teachers' explanation and students' imagination. Computers can display these dynamics directly. It helps students deepen their understanding more and enables them grasp geometric figures dynamically and enhance the power of conjecturing and proving geometric properties of figures, and promotes students to think geometric figures developmentally and discover new problems and/or properties by themselves. Examples are discovery of invariant property of circular angles and integration of the various cases of circular angles.

(B) In teaching spatial figures, it is usual that after explanation using real materials and models, the teacher proceeds to use of sketches on chalkboards or in textbooks. But it is difficult for students to transfer from the model to sketches and to make a connection between them. Computer display of three dimensional figures plays a role of an intermediate medium connecting models to sketch. Using the computer, students are expected to image three dimensional figures and their operations easily and to foster rich spatial notions. Examples are construction of spatial figures by movement of plane figures and cut of spatial figures, e.g., cube, by a plane.

(C) Observing display of computer which pictures various situations and phenomena dynamically, students can grasp concrete images and notion of function. The computer can draw graphs of functions immediately when the equations of functions are given. Observation of graphs globally and locally through zooming in and zooming out enables students to realize important properties of graphs of functions and the interrelationship of graphs of various functions. Examples are introduction of the quadratic function defined as an increasing area of triangle in the square and graphs of inverse proportion and other functions.

(D) Computer can process a lot of statistical data and give various statistical measures and draw relevant graphs. This enables students to concentrate on statistical analysis saving many hours of tedious computation and data processing. Example is statistics by real data, such as students physical exercises, running, jumping and throwing. In probability learning, it is important to understand the concept of probability as a ratio of events which occurs repeatedly many times under the same condition. Computer simulation enables this experiment in a few minutes and contributes to students' building up probability concept. Sample survey is another application of the simulation function of computer.

We will see use of handheld calculators in secondary mathematics class here. The hand held calculator is used mainly for lightening students' burden of computation. For instance, in statistics students need to process and organize a lot of data and compute various ratios, handheld calculator does this work effectively. It is used when teacher emphasizes to proceed to a further objective utilizing results of computation, e.g., to evaluate areas and volumes of plane and solid figures. To calculate squares of approximate decimal fractions in order to determine a square root of a number in a given decimal point is another example.

4. Computer Use for Mathematical Problem Solving

Now we will concentrate on computer use for mathematical problem solving, which is a main theme in this seminar. First, we will examine the role of computer use in mathematical problem solving, taking the phases of problem solving which are proposed by Prof. G. Polya into consideration, and attend to the posing and solving of a problem and their relation to computer use. Then we will look at the mathematics classroom and its climate in Japanese schools, as computer use is done in mathematics classroom but not in an isolated environment. It is crucial in computer use for mathematical problem solving.

(1) Role of computer use in mathematical problem solving

In the mathematical problem solving process, the following four phases are given by Prof. G. Polya (Polya, 1973):

1. understanding the problem.
2. devising a plan.
3. carrying out the plan.
4. looking back.

These phases are surely reasonable. It is clear that problem solving behaviors do not necessarily proceed in the order of from 1 to 4. For instance, in the phase of devising a plan a solver often becomes aware of misunderstanding or shortage of understanding of the problem and returns to the phase of understanding; in the phase of carrying out the plan a solver often goes back to the planning and makes up a new plan when he/she notices insufficiency of the first plan. Thus, we consider the above four phases as appropriate labels of what a person does in the problem solving process.

Computer use possibly contributes to each phase of problem solving. As to understanding the problem, various functions of computer help students understand what is meant by the problem sentence. For instance, using graphics, the computer draws a picture representing problem sentence or make diagram expressing vital parts of the problem, which enables students to imagine what the problem means concretely. Numerical computation can be used to represent concrete cases of the problem, if necessary. Simulation under the problem condition can give insight into understanding the problem. Using the computer appropriately, students are expected to understand what is unknown, what are the data and what is the condition.

As to devising a plan, it is necessary to find a connection between the data and the unknown. This is difficult for students with little experience and few resources. The computer has the possibility to assist these students by presenting not only various strategies for problem solving but relevant mathematical facts including concepts and formulas. Further, it helps students by repre-senting on the display overtly and concretely what they have drawn mentally, and facilitates to try out what they wish to do. These promote students' thinking for devising a plan to develop easily.

In the phase of carrying out the plan, operations such as numerical computation, operation of algebraic expressions and construction of figures are needed. For common problems, these are surely within students' competency, but the computer assists students when this is not the case by its powerful functions. In some cases, verifying of what they have conjectured may play an important role. However, whether it is appropriate and possible to leave the computer to check each step of carrying out and to make clear the correctness of each step or not is left as a question.

In the final phase, looking back, computer use has much value. Looking back involves not only to check the result, the method and the argument in solution of the problem, but to try to apply the result and method to similar or other problems, i.e., to generalize the result and method. This leads to discover new problems and to solve them, which is most expected in computer use in mathematics education. Thus, computer use has potential to play a great role in mathematical problem solving.

In mathematical problem solving, we should attend to who poses the problem and what kind of problems are to be solved. Usually the teacher and/or textbook poses the problem in mathematics classes. In this case, teacher is a poser of problem and students are solvers, and the their roles are separated. Very often problems in this case are of closed-ended type, that is, the problem contains necessary and sufficient conditions to give a unique answer and allows only varied approaches. For this type of problem, the solution is the only goal for students. However, there often occurs situations in which it is difficult to determine the problem itself or to make clear the necessary and sufficient conditions to solve the problem, such as those in the real life. These lead to problems of so-called open- ended type and mathematical problem solving is not limited to the fore mentioned closed-ended type. It is important for students to pose a problem, to collect necessary data, to set an appropriate condition by themselves in a given problematic situation and to solve it. That is, it is desirable that the student is a problem poser and solver simultaneously. In addition, as revealed in the phase looking back in the above, after the solution is acquired, students are expected to generalize the result and method by examining them. Here, they are also assumed to be problem posers and solvers.

As described already, computer use is powerful in inquiry and discovery in mathematics and facilitates to make conjectures. Using the computer suitably, students can implement the first step of the inductive paradigm of computation--conjecture-- proof in all areas of mathematics and their applications. Thus, the computer will be expected to help students in posing problems. In particular, for mathematical modeling of real situations, which contain complicated and complex data to be processed, the computer is surely indispensable.

(2) Computer use for mathematical problem solving in Japanese classroom

Computer use is done in mathematics classroom but not in an isolated environment. We will look at the mathematics classroom in Japanese schools, where the classroom climate, which is not necessarily the same as in the US, is thought to be crucial.

First, we will see the climate of Japanese mathematics classroom, which is made clear by the US-Japan collaborative research on mathematical problem solving. Prof. J. P. Becker elucidated typical organization of Japanese mathematical solving lesson and pointed out that it consists of the following six parts. Greetings at the beginning and the end are left out:(ICTM 1988)

1. review previous day's problems or introduce problem solving topic.
2. understanding the problem.
3. problem solving by students, working in pairs or small groups.
4. comparing and discussing (students put proposed solutions on the chalkboard).
5. summing up by teacher.
6. exercise, 2-4 problems.

Japanese teachers have a tendency that whole class instruction is central and group lesson and individualized lesson are auxiliary and supplementary to the whole class lesson. In the above organization, 1, 2, 4 and 5 are usually done in whole class under teacher direction with question-and-answer between teacher and students. In this lesson organization, the emphasized part is 4. comparison and discussion and many minutes in the lesson are devoted to this part. Cooperation, organization and orderliness are emphasized in a classroom and elaboration of solution by discussion and negotiation in the classroom are most stressed. Not only the answer, but the way of thinking to find the answer including strategies and approaches are discussed in order to generalize the solution of the problem of the day to a wider context. More stressed is the process by which a problem is worked out rather than the answer itself. It is a reflection in the sense of Prof. J. W. Stigler. (Stigler et al., 1888, Miwa, 1991)

Computer use will be subject to the climate of Japanese mathematics classrooms, that is, the computer is to be used mainly in whole class lessons; in particular, when the number of computers is very limited. It shows a tendency that often computer use be confined to whole class presentation of the problem situation in the introduction and the stage of understanding the problem. For instance, drawing a picture or making a diagram representing what problem sentence means is common computer use today. Following this computer display, the teacher expects questions and answers between teacher and students and between students and whole class discussion, if necessary. It should be noticed that the above drawing or representing realize a high degree of flexibility and give dynamic pictures which the computer allows, but not be reduced to a substitute of a chalkboard. When students are able to use the computer, it is used fully in the stage of problem solving.

Computer use for mathematical problem solving in classroom should not be confined to the above but be effective throughout the lesson. For instance, in understanding the problem, difficult points in the problem vary from student to student, and individualized study will be required. The computer allows the possibility for individualization. It is also true in problem solving by students. In the important stage of comparison and discussion of students' proposed solutions, when LAN of computers in a classroom are available, they can communicate freely what they did or did not do through the network. For these to be realized in mathematics classes, needed are full equipment of computers and change of the instruction paradigm and form.

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Discussion of Professor Miwa's paper:

Becker: Thank you, Professor Miwa, for that very thoughtful and interesting presentation, and thank you also for expressing yourself so clearly in English. It was easy for us to understand. Now we have some time for discussion. Are there any reactions or questions?

Demana: Yes, in your chart Professor Miwa, you mentioned about the computers in the schools but you didn't say anything about computers in Japanese homes. To what extent do Japanese homes have computers for their children to work with?

Miwa: We haven't yet thought a great deal about computer use in homes. But certainly there are a lot of homes with computers and a lot of commercial software on drill and practice is available. But our primary concern right now is with school education and, therefore, attention to computer use at home will come later.

Choate: In your talk, you made reference to the number of teachers who can teach computers. Could you explain what you mean by teach computers?

Miwa: This involves two things. Number one is how to operate the computer, what the computer is. Number two is teaching something using a computer. In my talk, I am thinking in both ways. Especially, in upper secondary schools how to operate the computer is more important, but at the lower secondary and elementary levels, using the computer in teaching something receives more attention.

Teague: I'm not sure to whom I should address this question, but thinking about the talk by Professor Miwa and Professor Fujii earlier today and your talk now, I'm curious what responses students may come up with when you pose those five problems if they had had a computer terminal in front of them, possibly with a spreadsheet or a graphics package. How would their responses differ? The next time you do a study like this, is that something that you will consider doing?

Miwa: Well, this is a difficult problem because there are at least two things to consider. One is that the computer exists and you can use the computer to solve problems. Another one is when you solve the problem you may be able to use the computer, but without even knowing about computer use, a student should be able to solve the problem. So there are these two considerations and, at this stage, we are not dealing with the first part yet, so I cannot answer your question further.

Ferrio: The new curriculum places a big emphasis on the use of computers. Could you speak a little bit about the goals for the number or the ratio of students per computer at the different grade levels, and how soon you expect to achieve those goals across the country?

Miwa: As far as the number of computers is concerned, in two or three years we are expecting to provide three hundred thousand computers to schools throughout Japan, which figures to an average of 22 to 23 machines per junior and senior high school class, and for elementary several. When we talk about computer use by students now, the focus is mainly at the high school level. In our high school (upper secondary school), there are three math courses, Math A, B, and C. All of them are electives - students may choose the course they want. Computers may be used in Math A; but Math A has four main parts, and students are supposed to learn two of them. Therefore, since the computer represents just one of the four, even if students take the Math A they may not study computer use. So, right now there is no reliable estimate of how many students will need computers and, therefore, even if you have computers for use, it may not be useful.

Zilliox: Will the computers be shared with other subject areas? Or is it just mathematics? And, what is the anticipated set-up? Will they be in a lab in one room together? When I think of twenty-five in a school, I'm trying to see how they're going to be spread out, all in one room or in individual rooms, just some picture of what it might look like.

Miwa: First of all, the computer is supposed to be shared by all subject areas, not just for math. If you have a lot of computers then we may have one computer lab with all of them, but usually there will not be a large number and so the computers may be moved to the class where they will be used. Also, the number of students in the elementary and junior high schools is getting less and that means that many schools may have rooms available, so there is a possibility to get the computers there. That's the present situation.

Morimoto: Well, especially at the junior high school level, many schools have more than twenty computers and, in that case, they are usually kept in one room and use a local area network system. At the elementary school level, usually the number of computers is very small and they don't have a local area network system, But right now there are a lot of opinions among the teachers and some say they should be kept in one place and some say they should be shared. So, there is no consensus yet.

Sawada: In the 1990 statistical data gathered by the Ministry of Education, one of the questions asked was where do you usually keep the computer in the school? The number one response in the elementary and junior high schools is that they are kept in a property room. For high school, they are kept either in a special room or in a computer lab. I will talk about this Wednesday during my presentation.

Becker: One last comment or question. Mr. Morimoto.

Morimoto: Usually when you install computers in the school, there are two stages. The first installment contains only several computers; in this case, they are usually kept in a property room. But at the second stage, the second installment, usually a lot more are purchased and they are kept in a special room or lab.

Becker: As we close this session now, there are several other important questions that I think Professor Miwa touched on that would be very good for discussion, but we'll have to discuss these over coffee or maybe at some other time. One is the question, of course, of teacher education. That's very important. Another question that occurred to me is the amount of time to prepare for implementing computers as part of the new syllabus. The new syllabus is completed but there will be some years yet before full implementation is accomplished. I don't know the details of that. A third thing, an impression I have is that at least at some of the grade levels there seems to be a rather careful and deliberate introduction of computer use; for example, Professor Miwa made reference to the syllabus of specific topics for which the computer would be used with students. And then, finally, another which, for me, is a very important characteristic of the use of computers in the classroom is a general theme of the need for very careful management of the class in using computers. This is another prominent characteristic of integrating the use of computers into the mathematics curriculum.

Thank you and as we finish this session, I want to introduce Professor Neil Pateman who is a mathematics educator on the faculty of the University of Hawaii. It will be possible for him to join us from time to time in our deliberations and we welcome him. Also, last night at the reception I forgot to mention that Professor Nancy Whitman is on her way. She is coming from the mainland and will join us tomorrow. So, once again, Professor Miwa, we would like to thank you for an excellent and very informative presentation.

Miwa: Thank you very much. I am very happy with our discussion.

Becker: Professor Uetake will now Preside for the next session. The coffee break will not come until after the next talk and discussion. Let's take a couple of minutes to stand and stretch, and then we'll begin the next presentation. As an aside, Professor Wilson just pointed out to me a very nice source of information and activities about arithmogons. I was not aware of it. It's in a book called Bottlecap Mathematics. The booklet is full of problems with some interesting arithmogon activities, published in the 1960's by Creative Publications. Professor Wilson has a copy and it sounds interesting. Now I'll turn the program over to Professor Uetake.

End of Discussion

CALCULATORS, COMPUTERS, AND ALGEBRA IN SECONDARY SCHOOL MATHEMATICS

James Fey

University of Maryland

Algebra is the most prominent strand in the secondary school mathematics curriculum of American schools. Students in college preparatory programs study algebra for at least two full years. Middle school mathematics programs include considerable work intended as preparation for algebra, and algebraic concepts and techniques play important roles in the other major strands of secondary school mathematics, like geometry, trigonometry, and analysis. Recent educational policy studies have suggested that success in algebra is a critical factor in students' academic progress beyond secondary school, so there is enormous parental and community pressure to assure that the mathematics programs of nearly all students include study of algebra.

Despite the prestige and time devoted to algebra in secondary school, it is, by many measures, a troubled subject. The failure rate in algebra is probably highest of any course in the high school curriculum, and university mathematics faculties regularly complain about the poor algebraic skills of their incoming students. Many students fail to master the array of symbol manipulation procedures for transforming algebraic expressions into equivalent form and for solving algebraic equations and inequalities, the use of algebraic methods to solve practical and puzzle problems, and the use of algebraic reasoning to construct mathematical arguments. Furthermore, the pressure to make algebra a subject learned by all students is greeted by skeptical teachers who believe that many students are ill-prepared for or simply incapable of learning "real algebra."

Given these background conditions, it should not be surprising that many mathematics educators have looked hopefully to calculator and computer technology in search of effective new tools for algebra instruction. There are, of course, some obvious attractive opportunities. Computers and calculators can be programmed to execute a vast array of algorithmic procedures for reading, storing, manipulating, and displaying numerical, graphic, symbolic, and logical information. Spreadsheets and data bases are powerful tools for constructing and analyzing patterns in numerical data; graphics programs give visual images of complex quantitative relationships; symbol manipulation programs perform formal operations that are the core of standard algebra courses—like factoring, expanding, simplifying, and solving. Furthermore, the fact that computers are programmable machines makes them flexible tools for solving new problems by construction of new algorithms.

Over the past two decades these computer capabilities have been applied to secondary school algebra in ways that can be sorted into three broad categories:

Computer as Teacher

Many who work in mathematics education see the improvement of teaching as the most serious problem and the most promising opportunity. When this concern is combined with interest in technology, the response is quite often some sort of effort to use a computer to perform teaching operations—to program the machine to execute ideal teaching behavior.

Some of the first algebra software developed for microcomputers was of this sort. Early electronic flash card programs provided practice problems covering the range of manipulative operations in a traditional algebra course. While this kind of software has been sharply criticized, it does offer at least three features that make it very attractive:

- Individual students can get problem challenges targeted on their personal learning needs.
- The computer can wait patiently for student responses to problems and then provide immediate personalized feedback on the student work, in a way that no single teacher can manage with a large class of very different students.
- Many students find interaction with computer software to be free of the anxiety that often accompanies work in front of another student or the teacher. There is no embarrassment in making errors "in front of a computer program."

Of course, skillful teachers can do more than assign algebra exercises and mark student work correct or in error. A perceptive teacher can diagnose student errors and give helpful hints through constructive dialogue with the student. More recent work in development of algebra software has tried to create "intelligent tutors" that provide useful diagnosis and remedial guidance as students work on practice problems. This challenge has been especially attractive to cognitive psychologists, including Dave McArthur and Matt Lewis at the Rand Corporation and Jill Larkin and John Anderson at Carnegie-Mellon University. The initial efforts toward intelligent tutors in algebra focused on ways to train students in the syntactic and procedural rules for writing and manipulating algebraic expressions. The tutor programs include underlying computer algebra systems that perform the symbolic manipulations being taught, schematic displays that help students organize their work on symbolic expressions with devices like expression trees and reasoning trees, and a variety of kinds of coaching advice available to students at various stages in their work.

Outside of the cognitive science community, the intelligent tutor programs aimed at teaching symbolic manipulation have attracted very little attention and there is only modest research evidence that the programs are effective. Many mathematics educators have complained that the symbol manipulation tutors take great pains to teach precisely those aspects of algebra made obsolete by the machines used for tutoring. In response to such criticisms, the authors of algebra tutors have recently turned to the more difficult task of building intelligent programs that help students in problem solving—especially in the phase of problem solving where verbally stated conditions must be translated into suitable symbolic forms.

The Carnegie-Mellon and Rand Corporation research groups have each developed such word problem tutors. The tutors typically offer students a screen with windows for displaying data and relationships in symbolic, numerical, graphic, and visual form. They offer a range of reasonable action choices and give feedback on the advisability or correctness of student decisions. There is only limited evidence on the effects of such word problem tutors, and most mathematics educators remain skeptical of the prospects. However, I am persuaded that the challenge of constructing such tutors provokes very deep analysis of the difficulties in connecting mathematical structures to the structures of problematic situations.

Computer-Based Exploratory Learning

Drill-and-practice and tutoring programs take a fairly direct approach to computer emulation of algebra teaching—the tasks and student actions are fairly narrowly constrained and the feedback is usually something like the words a teacher might use. There is, however, another family of instructional programs for algebra that give the student more freedom to explore structurally rich problem situations—very often in a game or contest against the computer or other students.

One of the first, and still the best, of this kind of instructional software for algebra is the collection of graphing programs authored by Sharon Dugdale and David Kibbey. Best known and often imitated is Green Globs, a game in which students are presented with a coordinate grid and thirteen 'globs' which must be hit by graphs of student-entered equations. The game offers no direct instruction on equations or graphs. The only feedback that students receive is the computer graph of symbolic function rules entered by the students, the opportunity to replay a game with new ideas, and the opportunity to study successful efforts by other students. However, reports of Green Globs in action reveal striking instructional effects as students discover and share ideas about equations and their graphs.

Graphical Exploration — Green Globs is also representative of the most prominent feature in software for algebra instruction—the use of graphical images to enrich the meaning of algebraic expressions and operations. There are literally dozens of pieces of algebra software (and now handheld graphing calculators) designed to give students a mathematical environment that supports exploration of the connections between symbolic expressions and graphs. The typical graphing program allows users to enter rules for one or more functions, choose domain and range or scales for a viewing window, and then inspect the computer-drawn graphs. An available pointer can be moved to various points of the screen to read coordinates. Rescaling options allow the user to zoom in on a piece of the graph or zoom out for a more global view. On some pieces of software the user can easily manipulate the graph or the function rule parameters and observe the corresponding changes in symbolic and visual representations.

Typical instructional activities that use function graphers give students a family of related rules to display and directions to search for connections between the symbolic rules and the graphs. For example, in elementary algebra students might be asked to study linear rules of the form $y = mx + b$ or quadratics in the forms $y = x^2$, $y = ax^2$, $y = ax^2 + c$, and $y = ax^2 + bx + c$. In advanced algebra the functions might be exponentials, rational functions, or the varieties of periodic functions. While the goal of these activities is clearly to teach some mathematical ideas, the software itself is essentially a pedagogically-friendly mathematical tool. The software does not usually present learning tasks for the students and its only feedback is to faithfully produce the image that the student asked for.

The basic idea behind interest in graphing calculators and computer software is the belief that experience with visual images of algebraic expressions will make the abstract symbolic forms more meaningful and easier to understand. Most mathematicians are convinced that the visual way of thinking is best. But early experiences with graphing calculators and computer graphing software have convinced many of us that learning about graphs is not a trivial task. Any coordinate diagram displays only a small piece of the total graph of a function, and choice of computer window boundaries or axis scales is a critical prerequisite for producing a picture that will contain important and illuminating information. Furthermore, we are finding that unless students get very careful introduction to fundamental graphing concepts they tend to see the graphs as pictures of the physical phenomena being studied (the path of a baseball in flight or a car on the tracks of a roller coaster) rather than as displays of relations between numerical variables. Students frequently have difficulty giving reasonable numerical interpretations to the (x, y) coordinates of a function graph.

Numerical Exploration — To help overcome the difficulties students have with graph interpretation and to make better connections between the general statements of algebra and the numerical experiences of prior mathematics, most graphing software now comes in a package that includes options to produce tables of (input, output) values for functions. In many cases the table of numerical data can be displayed alongside the graphs and the symbolic expressions.

The importance of approaching algebra numerically has been most effectively demonstrated in the work of Frank Demana and Joan Leitzel (1988). Calculators can be used to clarify syntactic features of algebra like order of operations and to motivate and facilitate introduction of new number systems and operations like integers and exponentiation. Calculators and computer spreadsheets are especially helpful in giving a concrete foundation in numerical work for the concept of variable and relations among variables that are at the core of algebra. The symbolic expressions of algebra can be used to summarize patterns in number tables, and a spreadsheet display can be used to produce numerical instances of symbolic relationships.

For instance, the following type of spreadsheet display gives students special insight into the behavior of a quadratic expressions.

x	$3x^2$	$-10x$	4	$3x^2 - 10x + 4$
–5	75	50	4	129
–4	48	40	4	92
–3	27	30	4	61
–2	12	20	4	36
–1	3	10	4	17
0	0	0	4	4
1	3	–10	4	–3
2	12	–20	4	–4
3	27	–30	4	1
4	48	–40	4	12
5	75	–50	4	29
6	108	–60	4	52
7	147	–70	4	81
8	192	–80	4	116

Moves in both directions—table to rule and rule to spreadsheet—are proving very effective in making students comfortable with the abstract, and ultimately very powerful, language of algebra.

Symbolic Exploration — The natural purpose of computer-based graphic and numerical explorations is to give rich meaning to the symbolic forms of algebra. But some of the symbol manipulation software itself is being designed and/or used to support student explorations that lead to learning about rules for symbolic manipulation. For instance, some pieces of computer algebra software require, or at least permit, users to perform or plan steps in a procedure to solve an equation. Some of those computer algebra systems also allow the user to inspect the solution path taken by the program.

Computer algebra systems with SOLVE, SIMPLIFY, EXPAND, and FACTOR commands diminish the importance of user skill in those operations. But nearly everyone who has thought about the situation agrees that algebra students must still acquire some good sense about symbols—ability to predict the general form of results from symbolic calculations and to recognize insights that can be gained from expressing an algebraic relation in various equivalent forms. Computer symbol manipulation programs are being used for this purpose in several new curricula. For instance, to help students recognize the information readily available in factored form of a quadratic polynomial an algebra teacher might ask students to use their computer algebra program to factor, then graph, then table, and then find zeroes for the following quadratics:

$$x^2 + 5x + 6$$

$$x^2 - 3x - 18$$

$$x^2 + 9x + 20$$

$$2x^2 - x - 15$$

$$-x^2 + 5x + 6$$

etc.

Some of the special features of symbol manipulation software provide surprising and clever approaches to familiar algebraic topics. For instance, adventuresome students have used the Mathematics Exploration Toolkit to type in words like MISSISSIPPI, enter the simplify command, the get the exponential form

$$I^4 M P^2 S^4$$

Using multiple representation software, students are also guided to trace the steps in symbolic solution of equations or systems and to notice the effect of each step on the graphs representing the system. It is intriguing (but not always easy to interpret) the sequence of graphs associated with solving the linear equation

$$7x - 5 = 3x + 8$$

$$7x = 3x + 13$$

$$4x = 13$$

$$x = \frac{13}{4}$$

At this point we have very little solid research on effects of these activities. Since symbol manipulation software is only beginning to appear in handheld computers, there is much less use of those programs, but there is a group of very enthusiastic proponents for this style of calculator- and computer-enhanced exploratory learning.

Programming — When computers were first used to enhance teaching of algebra, students had to analyze given problems, design algorithms for their solution, encode the algorithms in a suitable language, and interpret the computer output. Students were commonly asked to write programs that would execute the numerical aspects of important formal procedures like solving equations. Proponents of this programming activity argued that it would help students to develop general problem solving ability and deepen their understanding of the mathematical ideas being analyzed for programming. Over the past twenty years many projects have tried to demonstrate positive effects of such programming in mathematics education (generally using some version of Basic). The yield from those studies has been disappointing, and right now it seems fair to say that few mathematics educators make algorithm design or programming a prominent part of secondary school or college curricula. That is not to say that programming is without possible benefit, but only that in U. S. schools programming is not an important part of computing in mathematics.

Computers as Problem Solving Tools

The array of numeric, graphic, symbolic, and multiple representation tools for teaching and learning algebra is truly impressive. Much of the new software is really very clever. But it is not hard for a teacher of the current school algebra curriculum to look at spreadsheets, function graphers, and symbol manipulation programs and then ask, 'How does that relate to what I am doing in my course?' It is clear that symbol manipulation programs like Derive, Calc, and Theorist relate to current school algebra curricula in much the same way that calculators relate to traditional skill-dominated elementary school arithmetic curricula. They lead one to ask, 'If the computer can do all these operations, why should students learn them?' But they don't offer an obvious way to improve teaching of the traditional skills to individual students. Even more puzzling is the relation of spreadsheets and function graphers to conventional algebra curricula.

As I have studied and worked on this puzzle over the past ten years I've come to believe that there are two keys to unlocking the potential of calculators and computers in algebra. Both follow from looking at calculators and computers primarily as tools for mathematical tools, not pedagogical tools or electronic teachers. First, we must focus on algebra as primarily the study of functions and their representations. Second, we must sharply reduce the current emphasis on algebra as a collection of procedures for manipulating formal symbolic expressions. Both of these recommendations address the fundamental curricular question 'What is algebra really all about?'

The case for focusing on function as the central idea of algebra is implicit in nearly all of the multiple representation software that displays functional relations among quantitative variables. That software is saying very clearly, 'If you want to think clearly about symbolic expressions, think about them as rules for functions." Of course, thinking with functions is central to progress in a broad array of topics in advanced mathematics. It is especially prominent in constructing mathematical models of interesting quantitative problem situations.

The case for de-emphasizing the symbol manipulation goals of current curricula is highlighted by the impressive capabilities of easy-to-use computer algebra systems. But the implications of reduction in traditional training are untested. Tony Ralston (1989) has argued that

> The single most important question in mathematics education research today is this: What is the correlation between being able to do mathematical symbol manipulations (arithmetic, algebraic, calculus, etc.) and the ability to understand the underlying mathematics in the sense of being able to apply it and to build on it to learn higher level mathematics?

The puzzles reflected in these proposals to redefine goals of school algebra have provoked a great deal of conjectural debate, many informal classroom experiments, and a few careful research projects. Many efforts only change the curriculum by adding student projects that require computer help with complex computations. However, a growing number of informal and carefully analyzed experiments are exploring the effects of radically different course goals and structures.

Three projects based at the University of Maryland—one in finite mathematics, a second in elementary calculus (Heid, 1988), and a third in elementary algebra (Lynch, Fischer, and Green, 1989)—illustrate more daring approaches. The next section of this paper describes some of what we've tried and what we've learned in the project which we call Computer-Intensive Algebra (CIA).

Computer-Intensive Algebra

The Computer-Intensive Algebra project set out to capitalize on the opportunities for improvement of algebra learning and, especially, problem solving that are provided by these new calculator and computer tools. With financial support from National Science Foundation grants to the University of Maryland and The Pennsylvania State University and with the cooperation of schools and teachers in Maryland, Pennsylvania, and Illinois we were able to develop a prototype for our new conceptions of the content and teaching in elementary algebra.

New Content Themes

Our new approach to algebra emphasizes three fundamental themes.

Algebra as a Source of Mathematical Models — Of all justifications for making algebra the core of secondary school mathematics, the most convincing is its contribution to problem solving in nearly every scientific discipline. Traditional algebra courses include a variety of application problems, but those notorious 'word problems' are really a poor representation of the ways that algebra can be helpful in reasoning about quantitative problems. The classical coin, mixture, age, and rate problems are usually rather formal exercises in translation from verbal to symbolic expressions, followed by manipulations that reveal one (or at most two) unknown numerical values.

A far more practical and powerful image of applied algebra is embodied in the contemporary concept of *mathematical modeling*. An algebraic model is an abstract representation of variables and relations involved in some quantitative problem or decision-making situation. The model might be expressed in symbolic form, using equations or formulas or inequalities; in graphic form; in ordinary language; or in a table of numerical values for the related variables. In working with a mathematical model to understand a situation it is common to answer a variety of significant questions of the 'What if?' form: What if problem conditions change? What if the goal changes? Models can be used to describe and understand a situation, as well as to find specific numerical values of variables.

If school algebra is to provide students with the understandings and skills required by realistic quantitative problem solving, it seems essential that the curriculum emphasize the global concept and component processes of mathematical modeling:

- Identification of variables and relations among them.
- Representation of the relations among variables in numerical, graphic, and symbolic forms.
- Drawing inferences about modeled relationships.
- Recognition of limitations in application of mathematical models to situations.

We believe this goal dictates substantial change in the treatment of algebra and its applications.

Variables and Functions — Looking at applications of algebra from a modeling point of view, one is struck by the need to make several changes in the way traditional topics are treated. First and foremost is the concept of *variable*. To students of traditional algebra curricula, a variable is a letter that stands in place of a definite but unknown number. The principal activity of algebra is finding x .

When algebra is used to provide symbolic models of quantitative relationships, the letters stand for quantities that really do change as situation conditions change. The important problems are not only finding a specific combination of values for those variables to satisfy an equation, but determining the effect of changes in one variable on the values of others or finding the value of one variable that produces maximum or minimum value for some other variable. Thus if students are to use variables and expressions effectively in constructing and reasoning about mathematical models, they must develop an understanding of variables that is richer and different from conventional curricula.

In mathematical models it is also nearly always the case that questions of interest involve relations among two or more variables. One or more input variables are used to predict the values of other output variables; the outputs are *functions* of the inputs. For students to become adept at working with such models they need a confident understanding of the function concept and its various representations. It is important that they be able to construct and interpret tables of input-output values and graphs and to recognize, from the form of an algebraic function rule, the pattern of numerical relationship modeled by that rule.

Effective mathematical modeling requires an ability to inspect scientific or economic data and choose a reasonable model for relations in that data. Conversely, when an algebraic model is proposed, the user of that model must have good intuition about the numerical and graphic form corresponding to that model. Thus students of the new functions/modeling algebra must be very familiar with the important families of elementary functions--polynomial, exponential, rational, and periodic.

We believe that the understandings and skills related to variables and functions used in algebraic models are also quite different and richer than what traditional courses, with their emphasis on formal rules for manipulating symbolic forms, provide. Furthermore, the tools provided by computer spreadsheets, function graphers, and symbol manipulation programs are extremely helpful in developing the new understandings and skills.

Conceptual and Procedural Knowledge — The proposed goals of teaching about variables, functions, and mathematical models are hardly controversial. However, it is quite reasonable to ask how an already full algebra curriculum can be revised to find time for the new and deeper conceptual and problem-solving objectives. The obvious target of opportunity is the vast amount of time now required to develop student skills in the many symbolic manipulations of traditional curricula.

Of course, any specific algebraic skill targeted for reduced emphasis in the curriculum will find defenders who illustrate its role in theoretical or practical science or in an aspect of advanced mathematics. Furthermore, many mathematics teachers are committed to the principle that proficiency in procedural skills is a prerequisite to understanding of basic concepts and problem solving strategies in any branch of mathematics. Despite these reasonable concerns about effects of changing skill priorities, it seems inevitable there will be a general drift in the direction of less emphasis on those manipulative skills that can be performed by computers and calculators. It seems a high gain/moderate risk project to test the possibilities.

It seems likely that an algebra curriculum which emphasizes variables, functions, and mathematical models--with computer-aided execution of required numeric, graphic, and symbolic manipulations--will permit students to study very difficult quantitative problems without following the traditional regimen of skill building. Furthermore, there is nothing about such a curriculum structure that prevents later development of formal symbolic reasoning skills. In fact, it is plausible that manipulative skills can develop much more effectively if based on rich numerical, graphic, and application experiences in algebra. What we need is prototype curricula that embody these features and test their feasibility.

Scope and Sequence of Topics — In our Computer-Intensive Algebra, the core of the program is a printed student textbook of nine chapters. Our program is intensive in its use of computers, but it is not computer-based in the sense of early computer adaptations of programed instruction.

From a mathematical point of view, our text is structured to develop student understanding of the mathematical modeling process; the use of numerical, graphic, and symbolic representations in building and studying models; and the families of elementary functions that are principal components of many significant models. The chapters and their recommended sequence of teaching are as follows:

1. **Variables and Functions** - This chapter introduces the fundamental concepts of variable and function through study of a rich array of problem situations in which quantitative variables *depend on* or are *functions of* each other. Students learn how to construct and interpret tables, graphs, and symbolic rules that model functional relations among variables.
2. **Calculators, Computers, and Functions** - This chapter introduces a succession of computer tools for use in studying functional relations among variables: Calculators for simple models, computer table and graphing programs for more complex functions and for questions that require a global view of the function, and computer symbol manipulation programs for solution of equations when successive approximation (numeric or graphic) is ineffective.
3. **Linear Functions** - This chapter examines the numeric, graphic, and symbolic properties of the important family of linear functions--relating slope, intercept, and rate of change to the symbolic form $f(x) = mx + b$. The goal is to have students able to identify linear patterns in data, graphs, or verbal conditions and to use their calculator and computer tools to answer questions about those patterns.
4. **Quadratic Functions** - This chapter examines the numeric, graphic, and

symbolic properties of quadratic and higher degree polynomials and the use of those polynomials (especially quadratics) in modeling relations among quantitative variables. The goal is to help students become adept at recognizing probable quadratic patterns in tables or graphs of data, to find rules to match those patterns, and to use various computer tools to analyze those patterns.

5. **Exponential Functions** - Survey and analysis of functions with rules of the form $f(x) = C a^x$ and their applications.

6. **Rational Functions** - Analysis of numeric, graphic, and symbolic properties of simple rational functions with their application to several important examples of inverse variation in scientific settings.

7. **Algebraic Systems** - This chapter examines use of computer numeric, graphic, and symbol manipulation tools to study systems of related variables. One part of the chapter examines systems involving several functions of a single input variable. The part of the chapter examines systems with several input and several output variables and the equations and inequalities that often model important questions in those systems.

8-9. **Symbolic Reasoning** - These chapters introduce the concepts of equivalent expressions and equivalent equations/inequalities and the formal reasoning processes by which such equivalence can be established. The goal of the chapters is to develop student ability to use formal reasoning as a complement to their explorations of computer-generated numeric, graphic, and symbolic patterns.

These chapter titles and explanations make clear the way that our CIA program places variables and functions at the heart of algebra.

In design of the text material we have been strongly influenced by our desire to implement several fundamental pedagogical themes. First, we wanted to have the key mathematical ideas and methods arise from encounters with problem situations that were as convincingly realistic as possible. Thus the first section of the first chapter, and nearly every other section of every other chapter, begins with a situation which students are asked to analyze. They are asked to identify key variables, relationships, and objectives in planning of a school talent show. In other chapters they begin the study of quadratic functions with several situations involving accelerated linear motion, the study of exponential functions with situations involving growth of populations and interest-earning bank accounts, and the study of rational functions with situations involving inverse variation examples of sound, light, and gravitational intensity. To develop understanding of specific and general aspects of mathematical modeling, we have included reasonably frequent activities in which students plan and conduct data collection to find relationships among variables.

This emphasis on teaching through situations is very much consistent with the emerging interest in situated cognition (Brown, Collins, and Duguid, 1989) and anchored instruction (Vanderbilt Group, 1990). However, we are not as pessimistic about the limits of mathematical learning to specific contexts (Perkins and Salomon, 1989). In fact, we make specific efforts in each chapter of our course to help students formulate concepts and meta-cognitive understandings that transcend the specific embodiments from which key mathematical ideas are formulated. As we have worked to formulate our new approach to pedagogy in algebra, we have consistently worried about the interplay between abstract mathematical ideas and representations and the examples of those ideas that are meaningful to young students.

Of course, our approach to instruction also includes frequent computer explorations-typically asking students to work in pairs at the computer to pose and answer many questions that are plausible in a rich problem setting. Computer graphic, numerical, and symbol manipulation programs are used both to solve problems and to discover general patterns in the subject.

New Instructional Themes

The algebraic capabilities of calculators and computers suggest some dramatic changes in goals for school algebra curricula. They also provide opportunities for change in predominant styles of teaching and learning (Heid, Sheets, and Matras; 1990). In a curriculum dominated by goals that focus on training students in execution of symbolic algorithms, the teacher's primary role is to demonstrate new procedures and to monitor student practice of those procedures, including review of homework. Despite all we know about the importance of active participation in learning, students play a largely passive role in the traditional classroom process--watching and then imitating the teacher's behavior.

New Roles for Teachers — In a curriculum that de-emphasizes training in routine procedural skills, that emphasizes computer-based exploration of algebraic concepts like variables, functions, and graphs, and that asks students to construct and study mathematical models of realistic quantitative problem situations, teachers must become adept at a variety of unfamiliar instructional roles.

In classroom discussions focused on construction and interpretation of mathematical models, the teacher must be a catalyst and facilitator of group discussion. Typical modeling situations have many interesting facets and students can often bring to analysis of those situations personal experiences and insights. The teacher must play a flexible role of guide for such discussions.

When students turn to a computer lab activity--either to explore some new algebraic concept or to study an algebraic model of some structurally rich problem situation--the teacher must also play a less controlling role than in traditional instruction. Students at separate computers will be making different discoveries and encountering a range of different problems. Teachers must circulate and provide stimulation and feedback to this array of activities. Then they must find some satisfying way to help the entire class reflect on their lab experiences to integrate their observations into coherent knowledge.

Both laboratory and extended problem solving activities present new challenges in management of classroom time. They tend not to fit so neatly into the prescribed time blocks of typical school schedules, and as students work at their own pace in a lab, the differences in their progress are frequently considerable. Day-to-day planning for this diversity is a real challenge.

As if the challenges of teaching in less structured discussion and laboratory settings were not enough, teachers of computer-intensive algebra programs must find new effective strategies for assessing student learning and assigning grades. If students are accustomed to working on extended problem solving activities based in a single modeling situation, it makes little sense to assess their learning with a sequence of unrelated skill items. If course goals can no longer be described by listing skills to be mastered, then traditional "percent correct" scales are questionable guides to grading.

All of these new roles pose a substantial challenge for teachers in computer-intensive algebra. However, they also promise an impressive reward of greater student participation in learning and an attractive new relation between teachers and students—as collaborators in study of mathematical problems.

New Roles for Students — The new roles for mathematics teachers imply and are implied by new roles for students as well. In any laboratory setting for learning students must adopt more self-directed strategies. This includes learning to cope with open-ended tasks, to make confident judgments about the quality of their understanding and problem solving, and to work effectively with others in a cooperative spirit.

In a curriculum that emphasizes conceptual learning and complex problem solving, students must adopt more reflective habits, looking for bigger pictures, and they must sustain their attention over longer time frames than those required by the typical skill-oriented lessons. They must make notes on patterns in their computer explorations and write significant reports on the results of their problem solving.

It seems unlikely that any teacher would be disappointed if students became proficient in these new classroom learning roles. However, it is also likely that such proficiency will not develop quickly or perfectly, and the impatient teacher will probably find frequent cause for dismay and return to the more tightly structured, teacher-controlled pattern of typical algebra instruction. We believe that would be unfortunate.

Production of a computer-intensive school algebra program that embodies the mathematical themes of modeling and functions, and that emphasizes conceptual rather than procedural knowledge, required development of new student texts, guides for teaching, tests of student achievement, and computer software environments. Our approaches to each of these tasks evolved over the years of the project, and we expect that other computer-intensive curricula in the future will take different forms from those we have produced.

Evidence of Effects

Since the algebra program we developed takes non-traditional curricular and pedagogical approaches to new kinds of goals for elementary algebra, we have been studying the effects of our material in a variety of ways. Through conventional and computer-based tests and individual student interviews we have probed the understandings and skills that students acquire. We have tried to compare those attainments to the algebraic ideas and skills of students in more conventional programs. We have made ethnographic and scheduled observations of CIA classes in action and have again compared the classroom interaction patterns of the computer-intensive classes with those of traditional classes.

Our data analysis from the most recent field trials is not yet finished. However, we have data from early field trials and from several targeted doctoral dissertation studies that give some hints about the likely picture. First, it is clear that the new focus on applications, modeling, and computer-aided learning and problem solving makes dramatic change in the content and interaction patterns of classrooms. In CIA classes there is more talk about applications, more attention to relating representations of ideas, and more student activity than in traditional skill oriented classes (Heid, Sheets, and Matras, 1990).

Second, it seems clear that CIA students acquire a much different and richer understanding of key concepts like variables and expressions or functions than do students in traditional algebra courses. They develop a more diverse repertoire of problem solving strategies and are less inclined to give up quickly in face of a problem for which they can't see an obvious solution path. Much of this flexibility in problem approach seems related to use of several different computer tools to perform important operations like solving equations and inequalities. Furthermore, CIA students seem more comfortable dealing with approximation in method and result-a striking change from the traditional image of mathematics as the most exact and exacting of subjects.

Finally, it appears that students who follow our exploratory, situation-based, computer-aided introduction to fundamental algebraic ideas are well-prepared to move to the next level of more abstract symbolic reasoning which is the starting point for traditional instruction. Students whose instruction emphasizes functions, modeling, and multiple computer representations of algebraic ideas do not invariably form perfect conceptions and skills in the subject. However, it looks to us as if the refocusing of algebra instruction made possible by computer tools permits students to work very much more effectively in solving the problems to which algebra is applicable.

Summary

My enthusiasm for developing flashy high-tech solutions to educational problems probably reflects another typically American trait. But American schools are very conservative institutions, and, as I suggested at the beginning of this paper, very little of the computer flash and dash is commonplace in mathematics classrooms at any level. Nonetheless, we are getting moving on the task of using computers as tools for teaching, learning, and problem solving in our mathematics classes.

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Discussion of Professor Fey's paper:

Uetake: Thank you Professor Fey for your thoughts on new and very interesting computer uses in school mathematics. Now for questions or comments. Who would like to begin?

Morimoto: I found your discussion very interesting. The presentation involved a move from made-up, artificial problems to real life problems and is an intriguing one. I want to ask if this approach has actually been tried out and, if so, what have the student reactions been and what sort of problems have arisen?

Fey: We've been developing and field testing material for the past six years and we've tried the material in several different kinds of schools with several different kinds of student populations. As we've gone along, we've learned how to do things better so it's not clear to me that we've yet reached the right answer or the final answer. But I think there are a number of indications of fairly striking success in getting students to think in this new way and, as Dan Teague was suggesting in an earlier question, how to use the technology to really solve problems. Students can develop an ability to apply these tools to solve problems, very interesting and more complex problems. But I don't think we've by any means got the final answer on it. There are many ways it can be done and what we've done could be improved.

Morimoto: Is there anyone else who has been doing this kind of research?

Fey: Yes. Dan Teague's school, at a different level in the curriculum, has been doing this kind of work and there are some other people who have tried things similar to it. Dan, do you know of other people who have? I think our project is the one that has a full and complete curriculum at this point. Judah Schwartz and his colleagues at Harvard have also been doing a lot of work in this area. Some are working along similar lines but they aren't quite as far along in terms of developing a full curriculum, to the best of my knowledge. And the things that Sharon Dugdale is going to talk about and the things that Frank Demana is going to talk about are very similar. There is a similarity of feature. I've talked about elementary algebra for beginning algebra students and they'll talk about similar ideas, perhaps at a more advanced level.

Nohda: Your talk was different in a couple respects from the paper handed out yesterday.

Fey: Yes. When I saw the papers that the other Americans had prepared, I tried to say less about some of the things that they were going to talk about, and somewhat more about some of the things in the latter part of my paper.

Nohda: I looked forward to the information from you. It was mentioned by someone earlier, during a discussion, that the term ill-structured was used regarding problems. In the Japanese presentations, the term open-ended has been used. Would it be reasonable to interpret these as being more or less synonymous terms?

Fey: I'm not sure that's the case. For example, I would call the problems from the 1980's from Japanese studies open-ended in that they can go many directions, but I wouldn't call them ill-structured. They have very clear structure to them. The problems I talked about are probably ill-structured in the sense that there isn't a clear question. There aren't clear criteria for deciding whether you've answered it or not. You could go many directions in forming the question, not just in finding the answer. So I think there is somewhat of a difference between the two.

Dugdale: In regard to Mr. Morimoto's question earlier, one difficulty that sometimes arises in integrating this kind of work into an algebra curriculum involves students who have learned to play the "mathematics in school game" very well. These students are very good at symbol manipulation, they have always made A's and consider themselves good mathematics students, and then the new curriculum confronts them with a change of what's acceptable and what's rewarded. Have you noticed that some of your previously most successful students are suddenly having difficulties with the new demands and that they resent it? What do you do about that? More generally, does this issue work itself out over a period of months, or does it persist as a problem?

Fey: Yes, we've noticed that sort of thing. With some students it persists if they are not able to handle the new demands. In many cases it works itself out over a period of time. I guess we try to keep working with the students to get them to believe that what they are learning, the new stuff, is important and that if they had the old skills but didn't have what we're trying to do, they've got nothing very useful. But I guess students are like all the rest of us in that they like to feel successful and it's harder to see that you've succeeded in this new kind of material. There aren't the clear benchmarks of progress, so I guess we keep working with the students and try to convince them that what they're doing is good and, in many cases, we succeed. Of course, there are cases where we don't succeed. There are cases where students finish the year believing that they've not gotten the best stuff.

Dugdale: On the bright side, do you see students who have not considered themselves very successful in mathematics who find the new approach more engaging and become more successful?

Fey: Absolutely. One of the most encouraging things about this work is to encounter students who have not been successful, but who now can be.

Uetake: The next question, Professor Teague.

Teague: I'll be speaking on Wednesday about this question that we are now addressing. We teach a precalculus course which follows two years of algebra and it is very difficult for some of these students who have thought of mathematics as memory (i.e., you remember mathematics...you don't think mathematically). Often times they have to struggle quite a lot to deal with the new ideas and to get over the feeling that asking them a problem they haven't seen before and expecting them to use what they know to create a solution, rather than asking questions we've taught them to do and asking do you remember how to do it. It takes great patience and there are some students who get, as Jim said, very excited about mathematics for the first time because they do get to think rather than just remember.

Becker: We're talking about using new technology in teaching mathematics and you're using a teaching method that is made possible by the introduction of the new technology. What things that students get out of this are the most important?

Fey: We have an opportunity to give students the feeling that the mathematics they are learning applies to real situations, believable situations. That's one of the first things. I think the second thing is, as Dan has suggested, students who don't find the traditional symbolic material at all attractive find that they can do mathematical work, that the mathematics is not limited to the ability to live in this very small world of specific rules. There's an opportunity for each student with different learning styles and different interests to contribute in mathematics class.

Hashimoto: Professor Becker made reference to teacher training earlier. Looking at examples like yours, in the example you gave, it seems that the training, abilities, and quality of the teacher would be extremely important. As you prepare to disseminate this material, what kind of training and related things do you find necessary for the teachers?

Fey: Well, many of the teachers who are now teaching in the traditional algebra course don't have the perspective about mathematics that we'd like them to have. We haven't disseminated this material very broadly yet. We've worked very closely with teachers. They've spent a lot of time asking questions and we've been there to support them very often. Rapid dissemination of this program would be a severe problem. My feeling is that what the main use of our material has to be in teacher education first and then, after some time when teachers have become comfortable with this whole point of view, with many students. I don't think it can be massively implemented quickly.

Kaida: My question relates to Mr. Hashimoto's question and has to do with evaluation. In terms of training your teachers to teach the material, as a classroom teacher I'm thinking about how are these students evaluated? Because it always comes to the point where we need to assign grades.

Fey: You'd like to have them evaluated on how they can tackle a very large problem as a project, but there are some intermediate strategies that you can use. Along with our teaching material, we have a package of sample tests that we've developed. The tests emphasize student understanding of graphical images; for instance, can you interpret mathematical results? Then along with every one of these tests we have a section in which students go to the computer and use it to solve problems. We actually have some experience with having students take tests at the computer, so it's not impossible.

Sawada: As a Japanese teacher I feel very envious of American teachers in that they can freely put a new curriculum together and into practice. Japanese teachers are constrained to operate within set guidelines and therefore the introduction of new methodologies is rather slow. I would like to ask to what extent must a technique have spread and penetrated the curriculum in the United States before one can say this technique is used in America, or that this is an American technique?

Fey: I think you know well that the apparent freedom in America is only that, "apparent" freedom, because there are many informal constraints that prevent the kind of innovation that I'm talking about here. In some ways, perhaps you in Japan have more opportunity to put something dramatically new in place than we do because you can say this is going to be it; whereas, we've got to convince the country that it's different and useful. I don't think I'll even try to answer the last part of the question.

Uetake: Our time is up and I would like all of you to join me in thanking Professor Fey. We are now going to take a short break...about twenty minutes.

End of Discussion

USING CONSTRUCTION PROGRAMS IN THE TEACHING OF GEOMETRY

Jonathan Choate
Groton School

1. Introduction

More than 12 years has passed since the first microcomputers began to arrive in our schools. Some excellent programs such as the Geometric Supposer, Geometry Grapher, the Geometer's Sketchpad are now available to be used in teaching geometry. These programs provide exciting opportunities for teaching geometry because they can be used to perform geometric experiments. Geometric figures can be constructed and analyzed, hypotheses formed and then tested on other figures. This process allows the user either to prove or disprove the hypotheses by counter-example or to find some useful information for proving them. This paper will briefly look at how computers are being used in the geometry classroom and then compare the three most commonly used construction programs. Each program has its strengths and weaknesses and the program appropriate for a given course will be determined by the goals of the teacher.

Although this paper will discuss only three programs, there are other types of software appropriate for the teaching and learning of geometry. Computational programs such as spreadsheets can be used to perform many different types of calculations. They are very useful when one studies the more computational aspects of the subject such as area and volume. Tutorial programs such as Sensei's Geometry can be used to supplement the material presented in a course and be put to good use by the students who need extra help. There are also programs that guide students in writing proofs. There are computer languages, such as LOGO, which feature turtle graphics and which can be used with great success in a geometry class. Finally, there exist programs which can bring new ideas into the classroom such as discoverForm, a graphic design tool which allows the user to create a design and then transform it using linear affine transformations and 3D Images which permits students to construct, measure, and transform a variety of three dimensional geometric shapes. There are, also, a variety of programs available for the study of fractal geometry at a level appropriate for secondary students.

2. Using Construction Programs

Construction programs can be used to present new concepts and to have students discover new results as a class, in small groups or individually. Here are three commonly used instructional settings to accomplish these goals. The method used is dependent on the equipment available.

-If only one computer and a large display device is available, the computer can be used to demonstrate concepts for the entire class. In a sense the computer is being used as an electronic blackboard.

-If a lab setup with many computers is available, students can do their own experiments. In the beginning, handouts defining the experiment are very helpful. If the students work in groups of two or three, each student can assume a different task: one can be recorder, another can type at the computer. Later on, students can also use the lab to discover results on their own. There are some geometry courses being currently taught in this mode.

-If you have enough computers available, students can do projects individually. This method can be used to give students an opportunity to enrich and extend what the course provides.

Teachers who have used construction program as a regular part of their teaching have found that students often discover things which were not part of the original lesson plan. By listening carefully, and by encouraging students to pursue their own conjectures, many teachers have enabled students to discover a lot of geometry on their own.* For an account of a course taught almost exclusively using a discovery method see Houde and Chazan's Using Computers in The Teaching of Geometry.

The design of the classroom environment is important and plays a major role in making the use of technology effective. A classroom design which has been used successfully at both the secondary and college level is shown in Figure 1. Students are seated in movable chairs which allow them to either sit at the central U-shaped main table or to work in groups on the computers behind them. The instructor has access to a computer and a projection device at the top of the U-shaped main table and can either project onto a screen hanging in the front of the room or onto one of the side walls. This setup is versatile and has many advantages. First, it is very easy to change instructional modes. A teacher can present material to the class as a whole and then have them do experiments on the computer easily and quickly, because the students only have to turn around and they are at the computers. Secondly, when the students are working at the computers, the teacher can easily see all the monitors at once and keep track of how the class is doing as a whole.

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*Much of the preceding appears in The Teachers Edition of Geometry as part of the Using Technology section. It is used with permission of Houghton Mifflin Company.

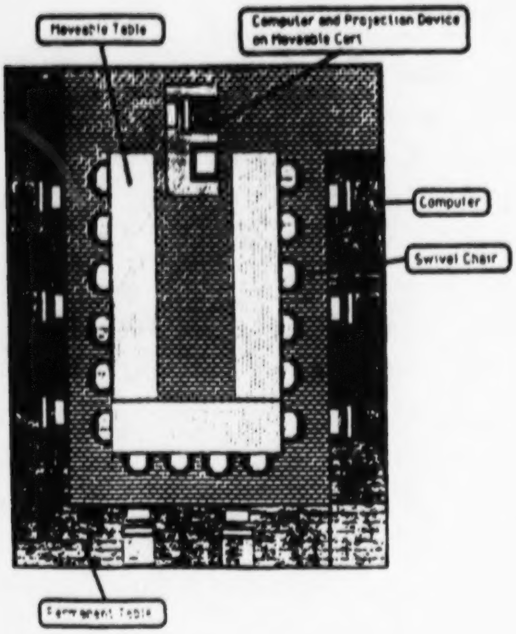


Figure 1. Classroom design for using technology

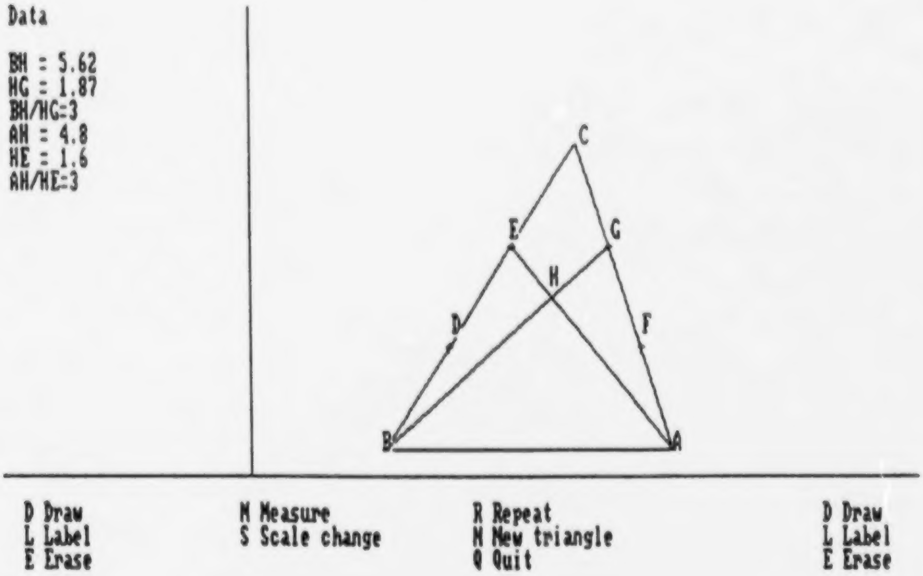
Likewise when the students are all gathered around the U-shaped table doing pencil and paper work, it is easy to see what each student is doing. Thirdly, the computer in the front of the room with the projection device can be used to guide the students through more difficult material. This design has been used to teach a wide range of secondary mathematical topics and is as effective for the teaching of algebra as it is for the teaching of calculus.

3. Construction Programs: A Comparison

The Geometric Supposer, Geometry Grapher and The Geometer's Sketchpad are three construction programs currently being used in secondary mathematics classroom across the United States. Although each program allows the user to construct a broad range of geometric figures and to engage in a variety of geometric activities, there are some fundamental differences. A brief description of the role each program was designed to play in the geometry curriculum is followed by a sample lesson plan illustrating how the software might be used. The lesson plans are designed to give the reader a sense of how each program operates and are not intended for classroom use. Writing precise step-by-step activity sheets for the use with construction programs is not easy because students will often experiment on their own and create constructions which differ from what was intended. Since all construction programs label points automatically, it is very difficult to know ahead of time what points will be labeled with what letters.

A. The Supposer series

The Geometric Supposer was the first geometry construction program to appear in classrooms. It is well designed and easy to use and is available for the Apple][series, IBM compatibles and the Macintosh. The series consists of four programs: The pre-Supposer, Supposer: Triangles, Supposer: Quadrilaterals, Supposer: Circles. The publisher has also made available extensive collections of activities. The pre-Supposer is designed for use in middle schools while the latter three programs were designed to be used as a part of a standard high school Euclidian geometry course in a variety of ways. The main purpose of the series is to give students a tool for constructing and measuring geometric figures so that they can come up with conjectures about the figures. This is accomplished by performing constructions on a figure, measuring appropriate lengths, angles and areas, calculating appropriate ratios and then repeating the process on a different figure of the same class. For example, after discovering that any two medians (the line which joins any vertex of a triangle to the midpoint of the opposite side) of a triangle intersect at a point such that the ratio of the distance from the vertex to the point to the distance from the point to the midpoint of the opposite side is 2:1, students might try to generalize the result by examining intersection of the lines joining the trisection points of the sides. The Supposer: Triangles could be used as follows. The Supposer is a menu driven program and all operations are defined by selecting options from menus which appear at the bottom of the screen. Figure 2. shows what the screen would look like after the following activity was completed.



```
Data

BH = 5.62
HG = 1.87
BH/HG=3
AH = 4.8
HE = 1.6
AH/HE=3
```

Figure 2. Supposer: Triangles screen

Activity 1: **Exploring the intersection of special lines inside a triangle.**

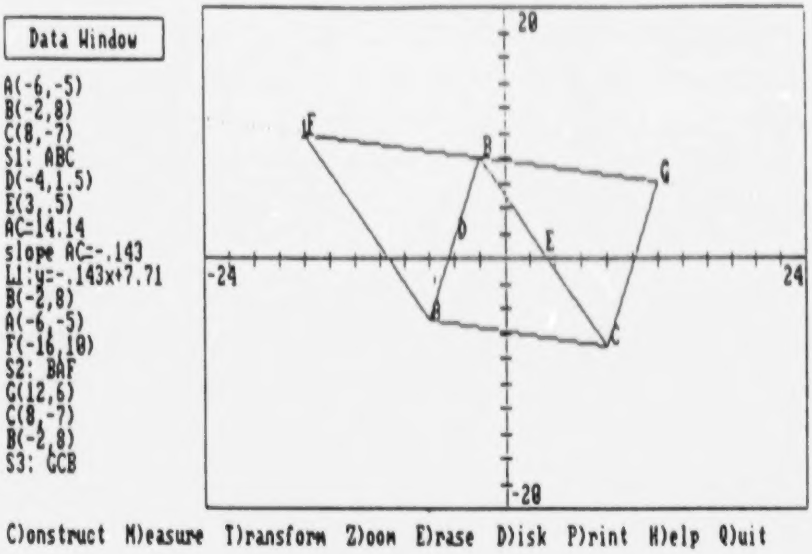
1- ***Select New from the main menu and construct a 7, 8, 9 triangle using the SSS option. A triangle ABC will appear on the screen. There is a scale option so that the triangle can be made larger or smaller.***
2- ***Construct a point E on BC such that CE/BC=1/3.***
3- ***Construct a point G on AC such that CG/AC=1/3.***
4- ***Construct segments BG and AE.***
5- ***Label point H, the point of intersection of the two segments BG and AE constructed in part 4.***
6- ***Use the Measure option to find the ratios AH/HE and BH/HG.***
7- ***Use the Repeat option to perform the same construction on a different triangle. Find equivalent ratios for this triangle.***
8- ***State a conjecture suggested by this activity.***

What makes the Supposer so useful is that after a construction is done once, it can then be repeated on any other triangle by using the Repeat option which will redo the entire construction automatically. Once the construction is complete, the Measure option can be used to collect data and calculate ratios which are all printed in a data window on the left hand side of the screen. Since a record of all the data gathered and ratios computed is present on the screen, it can be analyzed and conjectures made. In the preceding example, the ratio turns out to be 3:1 instead of 2:1. Many students will ask about what happens if you use quadrisection points, and they are well on their way to finding a general result. Once the students are convinced they have a result which holds true in a number of examples, they can start looking for a proof.

The Supposer does has have some drawbacks. First, you have to use separate programs for each type of figure you wish to study and your options are limited. There is no Supposer: Pentagons. Secondly, there is a limit on the size of the construction you can perform. You can not label more than 26 points. Finally and most importantly, it allows access to only some of the geometric tools which are now included in the secondary mathematics curriculum. Missing are any use of coordinate methods and transformations including translations, reflections, rotations and dilations. Since one is forced to use only the constructions known to Euclid, this is a good program for the study of classical Euclidian geometry but not for work in coordinate or transformation geometry.

B. Geometry Grapher

Geometry Grapher is a construction program which was developed several years after the Supposer and was designed to make up for two of its major deficiencies: lack of coordinate and transformation methods. It is produced by the Houghton Mifflin Company to supplement Geometry by Jurgenson, Brown and Jurgenson. Geometry Grapher, like the Supposer, is menu driven and all menus appear at the bottom of the screen. It is available for IBM compatibles and the Apple][series. Unlike the Supposer, Geometry Grapher does not constrain the user to one class of figures such as triangles. Points can be defined by coordinates, by finding intersections of lines and circles, by specifying a ratio in terms of other points on a line segment and by a movable cursor. Polygons, or shapes as they are called in the program, are defined by specifying vertices. For example, if points A, B, C, and D are defined then the shape ABCD is the polygon consisting of segments AB, BC, CD, and DA. Lines can be defined either by specifying a segment contained on the line or by equation. When defining a line by equation, one has a choice of using any of the three most common linear forms: slope-intercept, point-slope or standard form. Like the Supposer the program allows the user to collect data which is presented in a data window. Unlike the Supposer, once a point is defined, its coordinates are given and, once a line is defined its equation is displayed in the data window. The following activity has been used to help students come up with an alternative proof of the theorem which states that the sum of the angles of a triangle is 180 degrees. Figure 3. shows what the screen would look like after the following activity was completed.



```
Data Window

A(-6,-5)
B(-2,8)
C(8,-7)
S1: ABC
D(-4,1.5)
E(3,.5)
AC=14.14
slope AC=-.143
L1:y=-.143x+7.71
B(-2,8)
A(-6,-5)
F(-16,10)
S2: BAF
G(12,6)
C(8,-7)
B(-2,8)
S3: GCB
```

20
-24
24
-20
F
B
G
D
E
A
C
C)onstruct M)easure T)ransform Z)oom E)rase D)isk P)rint H)elp Q)uit

Figure 3. Geometry Grapher screen

Activity 2: Investigating the sum of the angles of a triangle.

1- Construct three points on the screen such that point A is in the third quadrant, point B is in the second quadrant and point C is in the fourth quadrant.

2- Define ABC as a shape. Measure segment BC.

3- Construct a line through B parallel to AC by pressing C for construct, L for lines, E for equation. Answer the prompts for inputting the equation of the line using the data printed for segment BC in the Data Window.

4- Construct the midpoint D of side AB by pressing C for construct, P for points, R for ratio and entering .5 when asked for the ratio of AD/AB.

5- Repeat the preceding step to construct he midpoint E of BC.

6- With D as center rotate triangle ABC through 180 degrees forming triangle BAF.

7- With E as center rotate triangle ABC through 180 degrees forming triangle GCB.

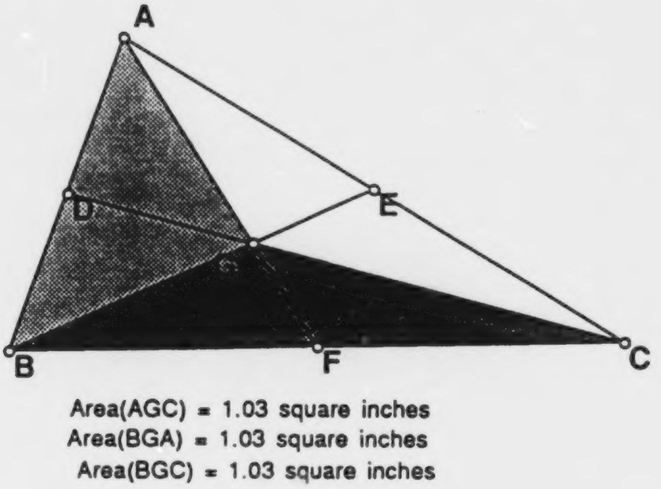
8- Use Measure to collect data about triangles ABC, BAF and GCB. Argue that triangle ABC is congruent to triangles BAD and CGA and that F, B and G are collinear. Use this figure to prove that the sum of the angles of triangle ABC is 180.

Several features distinguish Geometry Grapher from the Supposer programs. First, students can define geometric objects algebraically and use algebraic arguments to prove facts about geometric entities. For example, in the preceding example, lines were proven to be parallel by showing they had the same slope. Secondly, transformations are available and can be applied to any defined figure. This allows students to use transformations to create congruent and similar figures which can later be used in proofs. Finally, Geometry Grapher permits the use of 52 points, so more complicated constructions are possible than with the Supposer. It does not have the Supposer's Repeat feature which makes repeating the same construction on different figures time consuming, nor does it have the capability of creating dynamic diagrams as can be done with the Geometer's Sketchpad.

C. The Geometer's Sketchpad

The Geometer's Sketchpad was developed as part of the Visual Geometry Project at Swarthmore College and is currently being marketed by Key Curriculum Press, who has just published an innovative new book called Discovering Geometry. They will soon be publishing supplementary materials which will make use of the Geometer's Sketchpad.

The Sketchpad is a Macintosh program which takes full advantage of the Macintosh interface. All constructions are performed by clicking on objects on the screen and selecting appropriate operations from pull-down menus. It is easy to use and is the most powerful of the programs discussed in this paper. The following activity is designed to help students explore some of the properties of the centroid of a triangle. Figure 4. shows what the screen would look like after the following activity was completed. The relationship between the path of the vertex and the path of the centroid resulting is an interesting one and well worth investigating.



Area(AGC) = 1.03 square inches
Area(BGA) = 1.03 square inches
Area(BGC) = 1.03 square inches

Figure 4. Sketchpad screen

Activity 4: Investigating the areas of the three triangles formed within a triangle using the centroid as a vertex.

1. *Use the line segment and label tools to construct a triangle on your screen.*
2. *Using the mouse, highlight one of the sides by clicking on it. Pull the construct menu down and select Point at Midpoint. Repeat for the other two sides.*
3. *Select the line segment tool, and construct the line joining one of the vertices of your triangle to the midpoint of the opposite side. Repeat the preceding for the other two vertices. You have now constructed the three medians.*
4. *Click on one of the vertices, hold the button down and move it. As you move the mouse, the vertex you clicked on will be dragged around the screen and the rest of the figure will change accordingly. What do you notice about the three lines you constructed in the preceding step?*
5. *Select the pointer tool, click on one of the medians, hold the shift key down, select a second median, pull down the Construction Menu and select Point at Intersection. You have now constructed the Centroid of the triangle.*
6. *Select the pointer tool, click on the centroid and, while holding the shift key down, click on two other vertices. Pull the Options Menu down and Select Polygon Interior. The triangle formed by the centroid and the two vertices you selected should now be highlighted. Pull down the Measure Menu, select Area and the area of the triangle you just highlighted will appear in the upper left hand corner of the screen. There are two other triangles which can be constructed in a similar fashion. Construct them.*
7. *Click on one of the vertices of the triangle and drag it around the screen. What do you notice about the areas of the three triangles you just formed? State a conjecture.*

The Sketchpad has two other features which will allow students to play with geometric concepts in ways never possible before. The first of these is the Trace feature. In the preceding example, if one were to highlight the centroid, select Trace from the Display Menu and then drag any other vertex around the screen with the mouse, the path of the centroid will be plotted on the screen. Figure 5. was created by dragging vertex A along the line AH and doing a trace of the centroid which appears as a path of small heavy black circles.

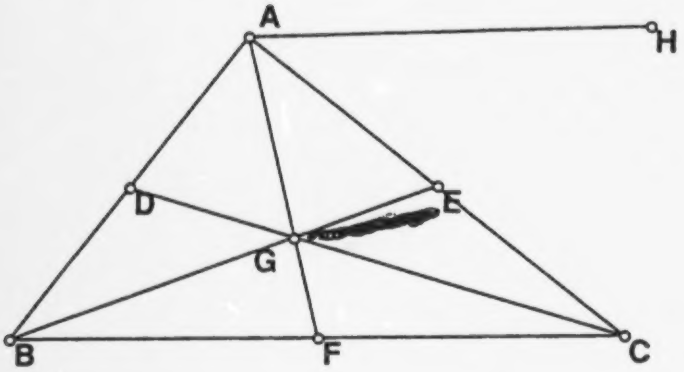


Figure 5. Trace figure

The Sketchpad has a second related feature. If one creates a line or circle on the screen and then constructs a point on that object, that point can then be made to move along the given shape using the Animate feature. Figure 6. was created by moving vertex A around a circle and plotting the trace of the centroid.

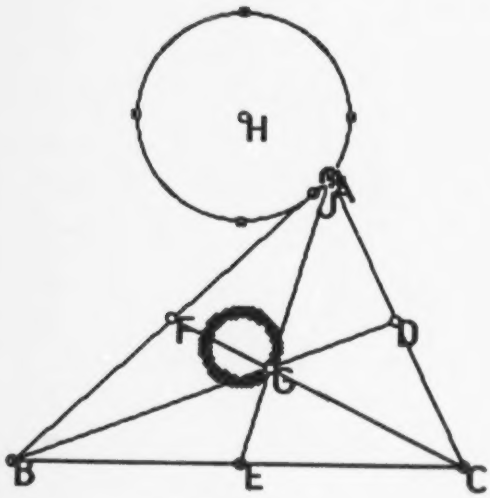


Figure 6. Circle created using the Trace feature

Trace and Animate introduce a new, dynamic aspect to the study of geometry and will permit teachers to easily pose problems never thought of before. These two features are unique to this program and are what differentiate it from Geometry Grapher and the Supposer. The Sketchpad allows students to construct geometric objects and to search for relationships in ways not possible until now. This is by far the most powerful program of the three and has set a new standard for future geometric construction programs. Programs like the Sketchpad will give students a new problem solving tool: the ability to build interactive models of problems. Consider the following problem which appears in many calculus books.

A four foot high picture hangs on a wall in an art gallery. The bottom of the picture hangs 2 feet above the eye level of a person viewing it. Where should the person stand so that the angle formed by the top of the picture, the person's eye and the bottom of the picture is a maximum?

Although this problem is often solved using calculus, it can be done using only elementary geometry. To arrive at a geometric solution, students must first play with the problem and get a feel for it and this is where the Sketchpad can be very useful. Figure 7. shows a Sketchpad diagram used to solve the problem. In the diagram, segment AC represents the picture and E_1 and E_2 represent two different viewing locations.

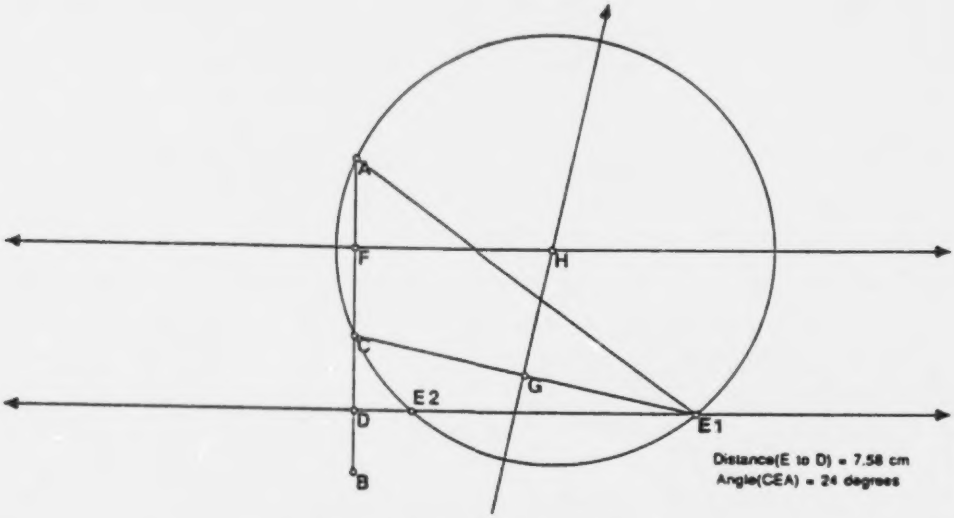


Figure 7. Art gallery problem diagram

Activity 4: Finding the best viewing angle.

1- Construct a vertical line segment AB on the left hand part of the screen.
2- Construct a point C on segment AB such that AC has length 4.
3- Construct a point D on CB such that CD has length 2.
4- Construct a line through D perpendicular to AB.
5- Construct a point E on the line constructed in part 4 to the right of segment AB.
6- Measure segment DE and angle AEC.
7- By dragging point E along the line constructed in part 4, collect data to fill in the table below.

Length of DE	angle AEC

8- Use your table to find an approximate solution to the problem.
9- To find the exact solution, notice that if *a* is some value less than the maximum value of angle AEC, then there are two distinct points E_1 and E_2 which will produce a viewing angle of *a* degrees. Argue that if angle AE_1C is equal to angle AE_2C then points A, C, E_1, and E_2 must lie on a circle.
10- Construct the circle described in part 9.
11- Drag point E along the line constructed in part 4 and watch what happens to the circle you constructed in part 10. Describe in words what is happening.
12- Use your observations in part 11 and what you know about circles to calculate the exact answer to the problem.

Conclusion

As the preceding example shows, there now exists software for use in the teaching of geometry which can change dramatically both how the subject is taught and studied in secondary schools. Technology exists which enables students to create, measure and transform geometric objects with ease and, hence, to actually become geometers. They can use construction programs to explore concepts, build models of problems, come up with conjectures and develop ideas for proofs. The role of teachers can change from that of distributor of knowledge to problem solving consultant. What software a teacher uses will depend on the goals for the course. If the course is to be a traditional one with a lot of emphasis on classical Euclidian geometry, the Supposer series would fit in very well. If the course is to take a more algebraic approach and employ geometric transformations, then Geometry Grapher will work well. If one wants a tool which will use transformations, allow students to play with problems, and to build dynamic geometric models, the Geometer's Sketchpad would be an excellent choice. In short, the technology now exists to teach geometry in new and exciting ways so that students in the coming years will know far more and will be better problem solvers. To achieve this goal will take a tremendous effort on the part of the mathematics education community since it will require a massive re-training of teachers. This re-training will have to focus on both how to use the technology and on how to make effective use of it in the classroom. This will require teachers to change what they are teaching and how they teach. The recently published National Council of Teachers of Mathematics Standards for both curriculum and instruction contain guidelines for this re-training effort and show how technology can change the teaching and learning of mathematics. For the Standards to be implemented, everyone in the mathematics education community must make teacher re-training their highest priority. If we don't, we will never take full advantage of all that technology has to offer and, as pointed out in the numerous reports which lead to the development of the Standards, the result could be a mathematically illiterate population.

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Discussion of Mr. Choate's paper:

Uetake: Thank you, Mr. Choate for this interesting presentation. Are there comments or questions?

Sakitani: Once you begin using this software, when you manipulate the figures which are limited in number, you don't go to an infinite number of possibilities; is this correct?

Choate: I'm not sure I understand the question. Are you talking about the number of points on the circle?

Sakitani: In general, as you play, with a number of repetitions, what you can do is necessarily limited and finite. At some point you must come to a desire to make a proof, and do you? Has it ever happened with your students?

Choate: That's precisely the question that I was addressing; that is, if you do this, you get that, why? Yes, we talk about reasoning why things happen and, where appropriate, we do use proof.

Sakitani: Well, even with Japanese students proof is a difficult part in teaching mathematics. Now, by using this software and after you prove a result, is there any evidence that proof making becomes easier for the students as a consequence?

Choate: Yes and no. First yes. I had two students who took a problem out of a mathematics magazine...a geometry proof. They made a proof by using the Supposer program and playing with/constructing the diagram enough times until they understood how to prove it. Success. I really don't know if, in the long run, students find proofs easier or prove things better. To be honest I don't care. My goal is to help students to do mathematics. They now become geometers. And I think that you don't have to prove everything to do that. May I give one more response? In the problem, I talked about the triangle and rotating it about its/the midpoints of the sides. Students very quickly see that that's a proof - that the sum of the angles of a triangle is 180 degrees. I think that's a good proof.

Nohda: I am going to give my presentation tomorrow afternoon and will expand on the relationship between conjecture and proof using the computer.

Shimizu: I find this software very interesting and fascinating, but I also believe that, in many places, using the compass and straightedge may be necessary. Are you doing that too with your students?

Choate: As a teacher, if I want to teach students constructions, I use transparencies frequently. And we do a lot of constructions on the software. For example, in bisecting (demonstrated) I have students use paper, not compass and ruler...only paper...it is all done with paper.

Teague: John, it seems to me looking at what you've done that one of the strengths of teaching this way isn't that students would necessarily be better at proving things, but that they would be better at making conjectures about what to prove. Do you find in subsequent courses that this is true?

Choate: Yes. Because we use the same approach with the other. But it's harder in other subjects to have students come up with conjectures. We do a lot in algebra using spreadsheet programs and graphing. We have students discover for themselves tests for finding maxima and minima of a function by looking on a spreadsheet at the table, looking at a column of differences. And they all know they are looking for where the differences go from being positive to being negative. They, in a sense, discover that in class, with no calculus and no limits.

Morimoto: I am wondering whether some of the software is rather difficult for students to use. Do you set up the situations for the students, and then let them use the moving part, or do you ask the students to do all of it themselves?

Choate: I ask the students to do it all on their own. The software I've used the most was the second one, the Geometry Grapher. And students learn to use it in twenty minutes, fairly quickly, and do not have problems with manipulation. The last one, the Sketchpad, I have not used with students yet. I suspect time will be required for students to learn it. But teachers that I know who have used it report that not a great deal of time is required to learn it. In my case, students at my school use Macintoshes all the time. They word process with Macintoshes, so they understand how to use the mouse and they understand moving and clicking.

Becker: As an observation, it just occurred to me, John, that on one occasion in Japan I observed a very nice open-ended problem solving lesson. The problem was a circle given with no center and the students were asked to find the center in as many different ways as they could. Of course, the way they approached it was by sketching things with paper and pencil and paper folding. And they came up with many different ways to find the center. I was thinking that maybe they could be using Sketchpad to carry out this sketching and folding; but now the question was raised, perhaps by Professor Sakitani, about the relationship between these activities and proof. In fact, that problem was used for over two or three class periods to lay the basis for a chapter on circles in which a lot of proof was to be done. I am wondering out loud whether Sketchpad may have some application here?

Uetake: Our time is nearly up. There is time for a final comment by Mr. Choate.

Choate: My final comment would be that we've only seen the tip of the iceberg in the use of software that's available now. If some of you are interested, I'll show you some other geometry programs that allow you to teach and have students play with ideas never before possible; in particular, with the notion of iterating linear transformations, the notion of an eigenvalue, and an eigonvector and what that means in a purely visual form without any formalism, but the idea of fixed points. I guess my point is that there is now stuff that we can teach that we never could teach before...very important ideas, because we now have the technology to do it. I think that, as teachers and educators, we really have to keep asking ourselves the question what should we be teaching? I guess my final comment would be, to me the most important question, what do we need to give people to be citizens of the world? I hope we ask that question and I hope that we create a mathematics program that allows students to understand exponential growth and topics like that because that's what our students, be they Japanese or American, are going to have to understand and deal with in order to get along in this world. We're going to be living in the information age. Thank you.

Uetake: Thank you, Mr. Choate. We will now close this final session for the day.

End of Discussion

ON THE USE OF SOFTWARE IN MATHEMATICS CLASSROOMS IN JAPAN

Yoshishige Sugiyama, Toshiko Kaji and Yoshinori Shimizu
Faculty of Education, Tokyo Gakugei University
Tokyo, Japan

The aim of this paper is to discuss the current state of the use of software in mathematics classrooms in Japan. For this purpose, we first present a classification of computer use in Japanese classrooms, according to the aims of use of software and to hardware environments, and then introduce some lessons we had observed. We ask for permission, however, because the selection of lessons is biased; these examples may not necessarily represent the typical use of software in Japanese classrooms. But we think these examples belong to excellent practices in Japan.

First of all, although Japan has been one of the most prolific countries in the production of computers, the computer had not come into wide use in Japanese schools until quite recently. This situation had various reasons and background as follows; the Japanese Government had not financed enough to equip schools with microcomputers; most teachers were short on experience of computer use and could not make use of microcomputers; there were few software on the market, and it is time-wasting and high-costing to get software of one's own making; teachers, generally speaking, did not realize the effectiveness of computer-based instruction.

Despite the situation described above, a new tendency has been emerging in recent years; quite recently, under the circumstances, the importance of computers in the information society is widely recognized. The Japanese Government has decided to budget the funds to equip schools with microcomputers and the rate of spread of microcomputer use is increasing. With the widespread interest in computer use in mathematics classrooms, some good software on the market is now available and teachers have begun to be interested in the use of software in mathematics classrooms. So, we anticipate a new situation in which we can see computers utilized by both teachers and students in many mathematics classrooms.

We are now on the way to the advanced stage. So far, the great majority of classrooms are equipped with one microcomputer, and few classrooms are equipped with many microcomputers and not so many Japanese teachers use software in Japan. So it should be noted that, in this paper, when we say "the current state of the use of software in mathematics classrooms in Japan", it is more precise to say "the current state of the use of software in mathematics classrooms equipped with computer(s) in Japan." In this situation with few computers in mathematics classrooms, most teachers having little experience in using computers, and teachers not being able to utilize software fully, computers are to be used in the Japanese traditional teaching style, or teaching a class as a whole.

The remainder of this paper deals with a classification of computer use in mathematics classrooms in Japan, with a focus on the purpose of computer use in classrooms, and introduction of some examples of lessons or software which will fit into each classification.

1 For Better Understanding (one unit/classroom)

Even if there is only one personal computer in a classroom, it helps students to understand subject matter and deepen their understanding of mathematics in teaching the class as a whole. It is more useful than an overhead projector as it is able to represent movement.

If software has been written specifically for mathematics classrooms (i.e., matching contents to the textbook page-by-page), a teacher can use it without difficulties, even if he or she has little experience with personal computers. Examples of software are as follows:

Ex.1 The formula of the area of a circle (New CAL)
*"New CAL" is software developed by Tokyo Shoseki Company
Ex.2 The sum of the exterior angles of a convex polygon (New CAL)
Ex.3 Proof of the Pythagorean Theorem (New CAL)

This software seems to be effective in the summative stage of a class activity, but these examples are also useful for presenting problems (or tasks) and for encouraging discovery of laws and characteristics in phenomena. It depends on the treatment of software.

2 For Presenting Problems and Problem Situations (one unit/classroom)

We can use a personal computer to present problems and problem situations for the whole class.

Ex.4 A problem to find a locus of a point (vertex of rectangle) (New CAL. See Appendix 1.)

The problem is to find a locus and the length of the path of a point, vertex B of a rectangle ABCD which rolls on a given line without slipping. (The software is also a sample of mathematical simulation.) Personal computers are used to present this problem situation. Presented repeatedly, students can form a mental image of the problem and understand visually the meaning of it.

This software is useful not only for understanding the situation, but also for finding the locus. In the problem solving activity, the teacher gives suggestions for its solution by presenting the same display repeatedly and by changing its presentation.

Personal computers are also useful to verify the results after solving the problem. After finding the locus, students can check their finding by the display.

Personal computers are used to give other tasks (to find locus of other vertices A and C).

Software like this can give opportunities for even an unskilled teacher to use the personal computer.

Computers in most Japanese schools, we suppose, are used in whole-class instruction, as in 1. 2 D Video shows an example of using personal computers in the class as a whole.

Other examples of software are as follows.

Ex.5 A point which is the same distance from 2 points A, B and is on the given line. (New CAL)

Ex.6 Apply the Midsegment Theorem (New CAL)
(The quadrilateral formed by joining the midpoints of the consecutive sides of another quadrilateral)

Ex.7 The locus of the midpoint of a chord of a circle which has one end fixed. (New CAL)

Ex.8 A locus of the midpoint of a segment which has a constant length. (New CAL)

Ex.9 The area of a figure made from two overlapping two congruent squares. (New CAL)

3 For presenting materials(data) to find laws and properties in a phenomenon

(one unit (computer) for the teacher and one for each two students)

We can use a personal computer in the class not only to present problems and give suggestions for solving them, but also to help the students discover laws and characteristics in phenomena properties by showing various situations or by changing conditions. This is because the computer can present immediate information on various conditions in the situation. By having pupils choose conditions or pose conditions, many examples to find a hypothesis can be shown. Also, many examples to correct the hypothesis or to verify it can be shown immediately. These are merits of using computers.

Ex.10 Billiard problems (How many times will the ball hit sides before it reaches a corner?) (New CAL)

Ex.11 Inscribed Angle Measure Theorem (New CAL)
The measure of an inscribed angle is a half of the measure of its intercepted arc (central angle).

Ex.12 Quadrilaterals inscribed in a circle (New CAL) (grade 9)
If a quadrilateral is inscribed in a circle, then its opposite angles are supplementary.

Ex.13 The Theorem of a tangent-chord angle (New CAL) (grade 9)
The measure of a tangent-chord angle is a half of the measure of its intercepted arc.

4 Simulations and Experiments

(one unit for the teacher and one for every one or two students)

Personal computers can be used for giving simulations of real phenomena and of trials. Problems requiring many trials for solution and phenomena spanning a large length of time can be shown quickly on the display.

Ex.14 A ball rolling down a slope (New CAL grade 9)
This is a simulation of a ball rolling down a slope. In an actual situation, we cannot stop the ball in the process, but on the display we can stop it at any moment and get data to find a law for the rolling down phenomenon.

Ex.15 Tossing dice
Actual dice-throwing is limited by time, but a computer can shorten the process.

In the four cases 1-4 above, the computer is used effectively, even though there is only one unit in the classroom. Of course, personal computers can be used similarly if there is one unit for many students or if each student has his own. If there are several computers in the classroom, they can be linked in a network and used as if they were one. Or, as in 3 and 4, they can be used separately, and thus increase efficiency. In the next case, however, there must be one computer for each one or two students, or efficiency is lost.

5 Tutorial instruction assisted by computer

As in programmed learning, students can learn mathematics individually through question and answer. Tutorial instruction seems not to spread in Japanese schools, because we cannot find good software for this purpose and Japanese teachers seem not to like such a learning style.

6 Exercises of calculations/Exercises of problem solving

A personal computer can be used as a data bank of problems (or tasks). If problems are ordered according to the levels of difficulties in a computer, the computer can give a student the same type problem for which he/she made a mistake. Students are thus trained in calculation skill. This type of utilization of a personal computer can be seen often in Japan.

The next example shows a class in which software for a calculation exercise was used in teaching a class as a whole.

Ex.16 Practice of exercises of solving simultaneous equations (See Appendix 2)

This class aimed at practice exercises of solving simultaneous equations after a lesson of solving simple simultaneous equations. In this class each student used a personal computer, but students in the classroom learned together with their classmates.

This class began with a review of solving simple simultaneous equations. Then the students exercised solving simultaneous equations individually using a personal computer. The teacher monitored students' activities and gave instruction individually through students' computers. This type would not be popular in Japanese schools. Japanese teachers prefer whole class activity, and will expect that computers must be used in this teaching style in Japan.

7 Bringing up computer literacy

Computers are also used for bringing up computer literacy. Students will have been accustomed to computer use through touching a keyboard. The next examples show classes aimed to accustom students to the computer in learning mathematics. (See Appendices 3 ~ 5)

Ex.17 Learning color painting (understanding of triangle)

One of the aims of this class was to help children understand that a triangle is a closed figure. For this purpose, if a child paints a non-closed triangle, then the painting color will leak out of the triangle on the display. This experience will give the child recognition that a triangle must be closed. In this activity, children learned to paint the color at the portion which they select. This is another aim of this class.

Ex.18 Drawing a regular dodecagon

Through reviewing of drawing a regular triangle, a square, and a regular pentagon, children found that the sum of the exterior angles is 360. Using this property, students solved the task to draw a dodecagon.

Ex.19 Drawing an unfolding figure of a triangular prism

After learning to program dot points on the display and to draw straight lines binding two points which pupils select, students had experience to draw unfolded figures of a box. The task of this class was to draw an unfolded figure of a regular triangular prism. In this activity, children must draw a regular triangle, but this drawing could not be done by appointing points. To solve this problem, children must become aware of the drawing method designating direction and distance which they learned before.

8 Use of Computers in Solving Problems

In this case, children and students themselves use the computer as a tool to solve problems. By using the computer, they can create various examples by themselves and discover characteristics about problems, like 3 and 4.

Students can use one of the powerful functions of a computer and process data quickly, to solve problems. So real numbers (dirty numbers) can be used in problem situations, and actual life problems can be used as classroom material.

For this purpose, we must have good problems to solve by using the powerful function of the computer. As yet, we have few such problems in Japan. So, our task is to develop many good problems.

Ex.20 Spirolateral (from "Atarashii sansu" No. 243)
Ex.21 Calculation of a square root of 2 up to 100 places
Ex.22 Computations impossible using a handheld calculator: 999 to the 4th, 8th, 16th, 32nd power and so on.
Ex.23 Finding prime factors of interesting numbers, as follows.
111 = 3 X 37, 1111 = 11 X 101, 11111 = 41 X 271,
12 = 2 X 2 X 3, 123 = 3 X 41, 1234 = 2 X 617, 12345 = 3 X 5 X 823,
123456 = ?
Ex.24 "Tower of Hanoi" (We saw a class lesson at a junior high school in Southern Illinois using this problem.)

APPENDIX 1 (Example 4)

The Locus of a Vertex of a Rolling Rectangle Without Slipping on a Given Line

(7th Grade Mathematics Class)

This is an example of the teacher using personal computers without difficulty to present problems and problem situations for the whole class by using software available on the market. The software is mathematics simulation for lower secondary school published by Tokyo Shoseki company. The sample class is the 7th grade of a public school in Tokyo. The main subject for this class is to find the path and its length of a vertex of a rectangle ABCD which rolls without slipping along a given line through one revolution.

First, in the class, personal computers are used to present this problem situation.

(The word "Rolling" is written on the blackboard.)

T : What do you call this?

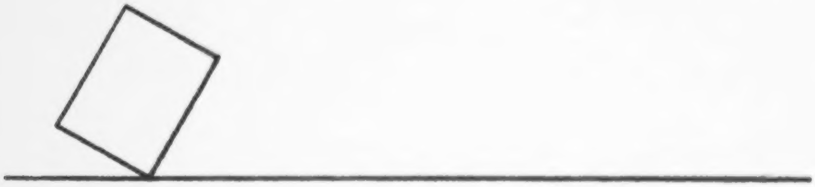
C(All) : Rolling

T: Today, there is a rectangle. What does rolling mean?

(The teacher draws a line and a vertically standing rectangle on the left hand at the end of the line and names the vertices A, B, C, D on the blackboard.)

T: Let's roll this rectangle along this line.

T: Look at your display. Let's roll it. Look at it carefully. (display 1 quickly)



T: I cannot mark the vertices A, B, C, D as this rectangle is rolling (on the display), but here is B. What path do you think the vertex B follows? When the rectangle is rolling, what path does the vertex follow? This is the line the rectangle followed as it rolled. Let's roll it, once more. Be careful. Follow the vertex B. Let's see it, once more (display 1). Once more (display 1). Finally, find the length of the path. Let's go. (Students are following the vertex B on the display eagerly.)

By presenting the display repeatedly in such a manner, students can form a mental image of the problem situation and understand the meaning of it visually. As personal computers can present the movement of figures actively, they can attract students' interest and attention, and motivate them at the beginning.

This software is also useful for better understanding the problem.

T: Here is the worksheet for you. Draw the path the vertex B follows on it. Read the problem. The problem is as follows.

The rectangle ABCD (AB=3cm, BC=4cm, AC=5cm) rolls without slipping along a given line through one revolution.

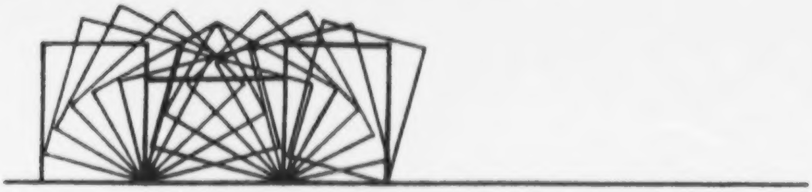
(1) Draw the path the vertex B follows.

(2) Find the length of the path the vertex B follows.

(3) Find the area of the part surrounded by the locus of the vertex B and a given line.

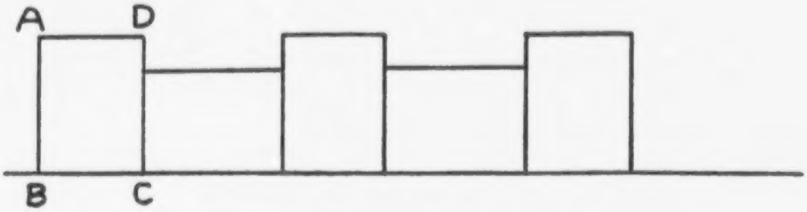
T: Look carefully at the path. The rectangle rolls slowly (display 1 slowly). Now vertex B lands again (display stands still). When it rolled like this, what path did the vertex B follow? Draw the path of the vertex B with the protractor and ruler.

T: Once more. Once more. When the rectangle went here, where did B go? From where to where did B go? B went this way. (Display was stopped midway through.) Up to here, where did B go? (Display was stopped midway through.) This is the end. For help, look at this picture (display 2: rolling and leaving a trace), look at this display, this arrow. B is here, isn't it? Here it is going up. It goes here. And where does the blue go? It goes here and there. After that, where is it? Think for yourselves.



Display 2

Then problem solving activity begins. The teacher will be able to give suggestions for its solution by presenting the same situation repeatedly in the process of problem solving and by changing its presentation. It is one advantage of computer use over manipulating the real object so that you can keep previous stages on the display. While the students work on paper, the teacher draws the rectangles on the blackboard.



(Figure 1)

And a student is asked to put the vertices A, B, C, D of the second rectangle on the blackboard. Same activity for the other rectangles. Then the teacher asks a student what path this B draws to this B.

T : Look at your display. (The teacher points to the display 2.) A yellow rectangle is rolling as a blue rectangle and goes there. There are so many lines that it looks confusing. The blue one rolls this way and comes here. Draw the path B moves along. How did it move? When you've finished, find the length of the path it traveled.

C : An arc of a circle is drawn.

T : Where is the center of the circle?

C : It is C.

Same question for the others. The teacher asks a student to find the length of the arc of the circle.

T: How long is the radius of this circle?

C: 3 cm

T: How long is the diameter of it?

C: 6 cm

T: How long is the circumference of it?

C: 6 π

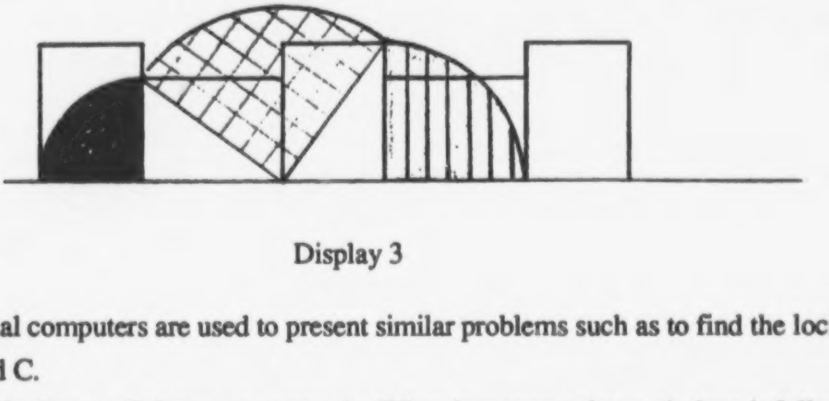
T: How much of the circumference of the circle with the diameter of 6cm does this occupy?

C: A quarter of it.

T: Let's divide 6π by 4. 6π / 4

Same question for the others.

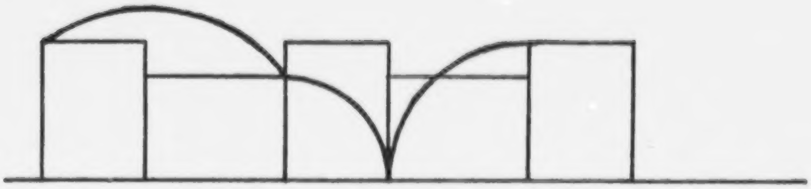
T: Look at your display (display 3). The problem is to find the area of the part surrounded by the path of the vertex B and a line. So, add the area of these triangles. (The teacher points to the part on the display 3). O.K., those who have understood everything so far, think about it. Let's try it. Do you understand the problem?



Display 3

Also personal computers are used to present similar problems such as to find the loci of the other vertices A and C.

T: This time we'll focus on vertex A. When it moves, what path does it follow OK? Draw the path of A. When vertex C moves, what path does it follow? Draw the path of C. Is everyone with me? Is there anyone who couldn't follow? Look at this display (display 4). Watch! This is A. OK. Try drawing it.

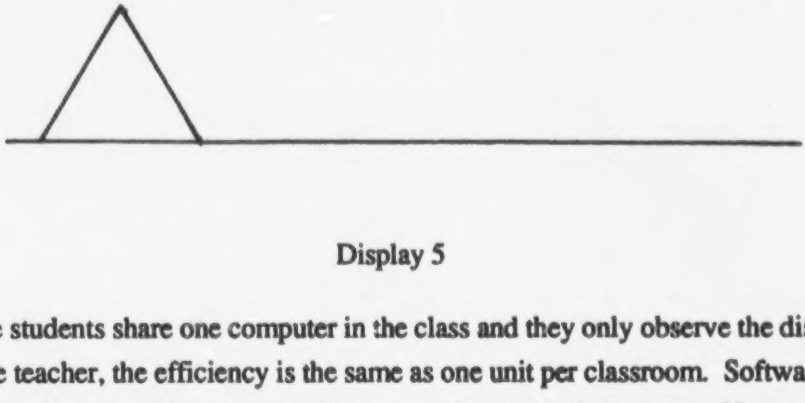


Display 4

Personal computers are also useful to verify the results after solving the problem. After finding the locus, students can check their finding by the display.

Finally, personal computers are used to present the extended problems such as to find the locus of the vertex of an equilateral triangle, instead of an rectangle.

T : Look at your display (display 5). This time this triangle rolls around. Let students think for themselves. This is homework.



Display 5

As three students share one computer in the class and they only observe the display presented by the teacher, the efficiency is the same as one unit per classroom. Software like this can give an inexperienced teacher opportunities to use the personal computer. If we will think out the way of use a little more, we will be able to use personal computers not only to present problems and problem situations, but also to give suggestions to solve them and to verify the result after solving the problems.

APPENDIX 2 (Example 16)

Simultaneous Equations (8th grade mathematics class)

This example is an eighth grade mathematics class in which personal computers are used as a data bank of tasks (for the practice of solving simultaneous equations of various types). In this class, each student used a personal computer, and whole class discussions were used to review the students' solution methods of simultaneous equations. The teacher monitored students' activities and gave suggestions individually through the computer network.

Review of Preceding Lesson

Teacher presents the tasks of the previous day on the display to review the solution methods for simple simultaneous equations.

$x + y = 3$ $3x + y = 7$ $2x + 3y = -6$
$x - y = 7$ $x + y = 3$ $x + y = -1$

Individual Work of Pupils

Next, new tasks are presented on the display.

1 $x + 5y = 14$ 2 $5x + 3y = 13$ 3 $3x + 7y = 16$ 4 The Other
$x - 2y = 0$ $3x - y = 5$ $4x + 5y = 17$

While each pupil engages the task, the teacher monitors their progress and difficulties they have using the computer network. If needed, the teacher can communicate with each pupil by the network.

Comparison and Discussion (Comparison and Discussion of proposed solution methods of simultaneous equations)

The teacher asks pupils to present the solution methods they used, with a focus on the equation 3. He names some pupils.

T: Explain how to solve the simultaneous equation 3, please.
C: I eliminated x first.
C: I eliminated y first.

Summing Up

T: We can use various methods to solve simultaneous equations. But today we have learned that it's better to select the most efficient method according to the types of simultaneous equations.

APPENDIX 3 (Example 17)

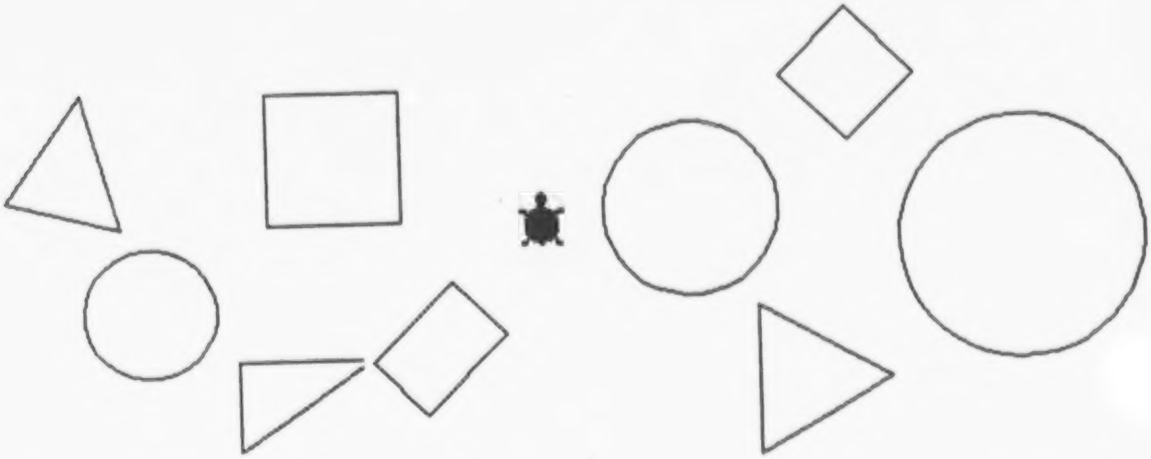
"IRONURI" Lesson: Is it a triangle? (1st grade mathematics class)

In elementary schools in Japan, computers are used for bringing up children's computer literacy. This is a first grade mathematics class, and one of the aims of this class was to help children understand that a triangle is a closed figure.

Using LOGO WRITER, children tried to color various figures including a triangle which had a "hole". When children tried to color this " triangle", whole display was painted because of its "hole". This experience gave them a recognition that a triangle is a closed figure. Through these activities, children learned how to use some keys to control the "turtle," some commands to color figures and, more importantly, they enjoyed touching the keyboard.

Introduction by Teacher

The teacher explains "Today's work", coloring various figures on the display. Pupils set the LOGO WRITER". They have already learned some keys to control the "turtle" and some commands in LOGO programming as follows: "set color", " fill". The teacher asks pupils how to color a figure (in order to review and confirm these commands).



[First Handout]

The teacher distributes to pupils handouts on which various figures such as circles, triangles, and squares are drawn. The teacher asks pupils to say the name of these figures, showing them on the blackboard.

First Attempt by Pupils

The display of the computer of each group shows the same figures as in the handouts. The teacher asks pupils to select the figure they prefer and to color it.

T: Do you find your turtle on the display?
Let's move it, by pushing the <F8> key. Then stop your turtle on the figure you want to color by pushing the <ESC> key. Now, let's color the figure. First command?

C: Set color

T: Next command?

C: Fill

Pupils are surprised to look at the colored figure. They try to color other figures, using the same procedure described above.

Discussion about the Happening

Suddenly, pupils in one group (group 8) raise their voice, being surprised, for their display is colored as a whole despite their "right" procedure.

T: Come together around group 8.

T: Although they tried to color a triangle, it's unsuccessful. They have colored the whole display. Why did it happen?

C: It protrude!

C: The triangle has a " hole".

T: Is it true?

Second Attempt by Pupils

T: Now, let's try the next task.

[The teacher distributes the next handouts the pupils on which various figures, different from previous one, are drawn. Those figures are also indicated on the display. Pupils try to color one of the figures on the display. The happening takes place on the display of group 3.]

C (in group 3): Oh, our display is now all in red!

T: What's happened?

C: It protrude!

C: We couldn't color the triangle.

Discussion

T: Why does it happen?

C: The triangle has a "hole".

T: Is it a triangle?

C : No, we can't call it a triangle because it has a gap.

T : This figure looks like a triangle, but isn't a triangle. So, let's select a triangle to color, looking at the display carefully.

Summing Up by Teacher

T : Now, could you color triangles? (Showing a copy of the figures) Here is a figure we could not color. Although this figure looks like a triangle, this is not a triangle. We cannot call it a triangle because the sides of it aren't linked.

APPENDIX 4 (Example 18)

Drawing Regular Polygons (5th grade mathematics class)

This example is a fifth grade mathematics class. In this class, to draw a dodecagon, pupils found a pattern of the LOGO program for drawing an equilateral, triangle, square, and regular pentagon.

Introduction and Review of Previous Day's Problem

Teacher distributes handouts to pupils.

T : Read the problem on it, please. [He names one pupil.]

C : " Let's draw a regular dodecagon, using the idea to write programs of drawing regular triangles, squares, and regular pentagons."

T : Okay, it's today's problem.

Pupils engage, first of all, to write programs of regular triangles, squares, and regular pentagons on the worksheets. Next, they input it by the keyboard to test if these programs work.

Regular triangle	repeat 3	[forward 100 right 120]
Square	repeat 4	[forward 50 right 90]
Regular pentagon	repeat 5	[forward 50 right 72]

The teacher asks pupils to present the program to draw each figure above. And he writes on the blackboard, what pupils said.

Finding a Pattern (Whole class discussion)

T : Can you find any rule or pattern in these three programs?

C : Triangle needs "repeat 3", square needs "repeat 4", and pentagon needs "repeat 5".
Thus, the number of times to repeat is equal to the number of sides.

T : Is there any other idea?

C : If we multiply the number after repeat" by the number after right", we always get 360.
[Teacher again writes on the blackboard what he said.]

"Today's Problem"

T : So, can you draw any polygons, which have various numbers of sides?

C : Yeah

T : Using that pattern we have just found, let's draw a regular polygon with 12 sides.

T : You'd better write the program on your worksheets first. And then confirm whether it works or not.

Pupils' Work in Small Groups

Pupils write programs on their worksheets, and then input it to confirm whether their programs work or not.

The teacher makes suggestion to the pupils who have done the work early, to color the polygon or to change the length of the edges.

Discussion of Proposed Program

Repeat 12 [forward 50 right 30]

T : Okay, is it done already?

C : Yes.

T : Show me your program, please.
[He names one pupil.]

C : Repeat 12 [forward 50 right 30]

T : Okay, did you try it on the computer?

C : Yes, it works.

Pupils' Work in Small Groups

T : Now let's try another polygon. Draw a regular decagon, as a next polygon.
[Pupils begin to work in small groups.]

Summing Up

T : Can you tell the way to write programs of drawing polygons?

C: In the case of a regular decagon, for example, because the product of the number after "repeat" and the number after "right" is 360, we will get the angle of the decagon by dividing 360 by 10.

Practice (Other Polygon) and Applications

After the discussion and the summing-up, the teacher encourages pupils to draw various figures using the rule. Some pupils draw polygons with 15 sides, 100 sides, and 360 edges. On other draws a figure like a flower, called "repeated figure".

APPENDIX 5 (Example 19)

Drawing the Development of Prisms (6th grade mathematics class)

This is a sixth grade mathematics class. In this class pupils tried to draw a development of a triangular prism. In the previous day's lesson, they had learned to draw a development of box using the "Set Position" command. So, they tried to use the same command to draw the equilateral triangle, which was a bottom of the triangular prism. But they could not draw the equilateral triangle by using "Set Position" command. To resolve this problem, they had to use the "Forward" and "Right" commands, which they learned a half year ago.

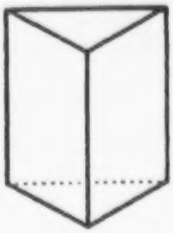
Introduction by Teacher

T: Today, I will show a solid none of you know. And I want you to draw a development of it. I wonder if it's pretty difficult.
[He carries a table to the center of the classroom and puts three triangular prisms on it.]

T: We call this solid a "regular triangular prism".
Can you tell the figure of the top and the bottom of this prism?

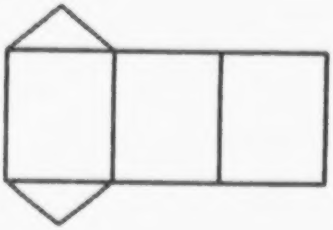
C: Regular triangle.

T: Can you draw a development of this prism?
[Teacher distributes handouts, on which a sketch of regular triangular prism is drawn, to pupils.]



Pupils' Work

First, pupils draw a rough sketch of the development of the prism on the "planning sheets", and then write a LOGO program. Looking at the display, they input it. These processes are done in small groups of three pupils.



The teacher goes from one pupil to the other, looking at their work.

Then the teacher takes a figure, drawn by one group, and copies it to distribute to all pupils.

T: Now, here is a drawing by Tanaka's group. Cut this figure out of the paper to confirm it's a correct development of the regular triangular prism. Does it go well?

[Pupils cut the figure out of the copy and construct the prism.]

C: Oh, it's unsuccessful.

The Problem Emerging

T: Can you explain why it didn't go well?

C: This triangle is not regular. It's an isosceles triangle.

T: Can you draw an equilateral triangle?

C: We can't draw a regular triangle on our sheets, because we can't plot the top point of the regular triangle.

C: Yes, because of that difficulty, we can't use the "Set Position" command.

T: Ah, you don't know how to do it. Discuss with your group members to resolve this difficulty.

A Solution of the Problem

T: Is there any good idea?

C: It's better to use the "forward" command instead of the "set position" command. Using the "Forward" command, we can resolve the difficulty.

[In the previous day's lesson, they had drawn a development of a quadrangular prism with the "set position" command. This drawing didn't need a "forward" command at all. So, they could not find a use for the "forward" command.]

T: Many of you blindly adhere to the "set position" command used in the preceding class. It's easy now to write the program for drawing a triangle. The difficulty is resolved, isn't it?

T: Can you draw an isosceles triangle?

Discussion of Proposed Program

T: Is it done already?

C: Yes

T: I will give you the copies of a figure, drawn by a group.

[Teacher distributes to pupils copies of a figure drawn by a group using "Forward" command.]

C: Cut it out to confirm whether or not the idea is successful.

[Pupils cut the figure out of the copy to construct the prism.]

C: It's successful.

Summing Up by Teacher

T: If we have some trouble in solving a problem, we must consider all of what we already learned to extricate ourselves out of the trouble.
Have your "head" be flexible so that we can recall quickly everything we learned.

Discussion of Professors Sugiyama, Kaji, and Shimizu's paper:

Dugdale: Thank you, Professor Sugiyama and Professor Kaji and Mr. Shimizu. We now have time for discussion and we will begin with a question by Professor Zilliox.

Zilliox: Who develops the ideas and the activities that teachers use in the classroom? Were they developed by teachers, the Ministry of Education, or commercial vendors? For example, the activities that we saw the teachers using in their classroom on the videotape. Were they ideas of teachers or did they come from the outside?

Sugiyama: A commercial software company produced the software, and they are just using it. The software is made by the company and each teacher is responsible to use it.

Teague: You've listed eight applications or uses of computers in the classroom. What percentage of use falls into each category?

Sugiyama: In this case, mostly they use the computer for developing computer literacy in high schools; secondly, it is used for presenting problems and problem situations, or for developing better understanding of mathematics in junior high schools and elementary schools.

Damarin: I'd like to follow up on the question asked by Joe Zilliox with a particular example. In the software showing the rolling rectangle or rolling quadrilateral, what part of that production of software did the teacher do? What I'm trying to get at is what was provided to the teacher to use as software, and then what did the teacher do with it? Was the entire demonstration provided or was some software utility package provided that the teacher then used to create that?

Sugiyama: Well, attached to the software are very simple manuals showing how to use it. But, in this case, the use of the software was mostly the teacher's idea. Teachers develop their own teaching method using the software.

Fey: I see in these lessons something that we heard about in the earlier problem solving discussion yesterday, that a class might have one problem that all the students work on for an entire class period. How do you see that students get mathematical principles and techniques out of this problem as opposed to learning how to solve that problem and how to solve another problem? Where do the mathematical themes come out, or are they just embodied by the problems?

Kaida: Could you clarify your question?

Fey: Well, it would be more typical in an American classroom for a teacher to show a mathematical technique applied to several different problems and the idea is that the focus of the lesson is on the mathematical technique and less on the problem. And what I see, maybe I'm getting a false impression, is a problem posed, and either it's an exercise in application of things that students already know or else I don't see how the underlying mathematical idea comes through in the solution of the problem. Does that make any more sense to you?

Sugiyama: Well, there are some cases when the teachers teach the solving technique first and then give problems; but, in general, in this case too, both the teachers and students attack this new problem together and, in the process of solving the problem and discussing it, they find out the mathematical topics and problem solving techniques.

Hashimoto: In general, when teachers make use of available software, of course they refer to the attached manual(s) and the teaching plan for the software; but in most cases, they develop their own lesson plans and then they use the software instead of following all the guidelines.

Ferrio: The activities that we saw in this presentation, are these experimental activities that are just being developed or are they in use in some classrooms in Japan, or are they even in wide use in Japan? I'm referring to the specific ones that we saw here?

Sugiyama: It has been said before that the level (frequency) of use of computers is still very low in Japan. In this case, this is one of the elementary schools in Urawa City which is recognized to be a very advanced school and they just demonstrated this. Of course, the software is available commercially.

Demana: It was hard to tell from the video how much the teacher was directing the lesson and how much the student was participating in developing the mathematics. Can you say something about what percentage of the actual flow of the mathematics was being done by the teacher, and how much of it the student was producing?

Sugiyama: Well, certainly students' participation is much longer than the teacher's explanation. I would like to give the example in the demonstration.

Demana: Well, the rolling rectangles problem, can you refer to that one?

Sugiyama: Well, in this particular case, probably the first five minutes was spent by the teacher explaining the problem and showing the demonstration using the computer monitor. Once the students understood the problem, they then are given about 25 minutes to work on the computer and try to solve the problem; in between time, the teacher can provide hints or suggestions through the monitor. And then about 20 minutes is spent in discussion and a summary, and the teacher might give some additional related problems too.

Becker: An observation I might make, in this connection, is that in many of the problem solving lessons we observed in Japanese classrooms, during the 20-25 minutes when the students were working on the problem, the teacher was very active, but not in speaking. The teacher walked around and watched what the students were doing and identified the things that he or she would want to emphasize later in the lesson, to bring to the attention of all the students for discussion. So, that kind of purposeful scanning of student work was a very important part of the lesson.

Fey: Related to this time flow in lessons, one of the things that we've found in computer lab lessons is that students may progress at very different paces as they try to solve the problem. Some students will solve problems very quickly and others won't get through the material. Time management is a very hard problem for American teachers in these kinds of open lessons and I wonder what techniques you use to avoid that problem.

Sugiyama: In the case of solving simultaneous equations, you saw it in the VTR video. Well, this is a special case because the teacher gave a set of many problems so that, within a given time period, the students could go at their own pace and fast students can do a lot more than slow ones. In general, how to manage students and individual progress within the whole class situation is a big problem and the Japanese are doing research on this now. But, in many cases the teacher prepares some additional challenging problems for those who advance more quickly.

Choate: A question about the rolling rectangle problem. What did the computer add to that lesson? As I watched the lesson I saw a lot of duplication of diagrams and drawings and I'm not really clear on why the computer was used in that lesson?

Sugiyama: Of course, it may possible to teach this material without using software. However, some students cannot visualize the process of the problem with only verbal expressions and that's why the software is used. Also, the demonstration shows not only the repetition of the same thing, but each time it's different. Some show the whole movement, while some show each process separately, and some show the trace or locus of the point (demonstration given). So, the first one is the locus of point A and the second one for point B…different.

Choate: Can I, as this is going on, ask a question? Is this software pre-packaged? Does the teacher come in and make that rectangle and write the program so that the teacher can make the rectangle roll, or is this a pre-packaged lesson?

Sugiyama: This is all pre-packaged. The teacher cannot change or do anything about the software as a whole.

Choate: So if I was to ask what would happen if the rectangle rolled around the circle, I could not use this software?

Sugiyama: In this case, you cannot make any changes or additions to the software.

J. Wilson: Following up on Jon's question, this is a problem that I've used in my problem solving workshops for teachers and with students for several years. We work it with a rectangle and a sheet of paper and everybody gets it down on their desk - and they do some rolling. I think Jon's question is, why not have kids doing these things with their hands? What is added by the software for this particular problem? Jon wants to change the problem. I've had kids say, what if we rolled it over to the corner and now we roll it up the side and we get a different pattern. There's certainly freedom in doing that. So, I think Jon's question and mine is, given an alternative way of doing this, where we would have children doing things with their hands and constructing these things in a realistic way, what does the software add?

Sugiyama: Well, just one example is shown here and, in this case, certainly the software problem is very simple and limited, but that means that the teacher needs no preparation at all - the teacher can just come and use it and then the teacher can give additional problems; and then, later, students can do the other kind of problem as you have suggested. In contrast, there is the software shown yesterday, the Geometer's Sketchpad. Stuff like that requires the teacher to spend a lot of time in preparing it. Of course, that kind of software has been developed in Japan too, but this, today, just shows a very simple case.

Wilson or Choate: My question is not necessarily whether or not I wouldn't use that software, but I am looking at the problem and I think there is , well, I have this orientation of several others here that the computer is a tool and when/if this problem is approached with the computer as a tool, for the kids to explore and look at the problem, then I could see that I would like to try some other things with it. If the software becomes a way of presenting the problem to students and it becomes essentially a way of delivering the problem, then I have less concern for it.

Sawada: In Japan there are many textbook and software companies. But, in general, teachers do not follow exactly what the textbook says or go along with the textbook and, therefore, most of the software is aligned with the textbook and these are the ones the teachers prefer to use. In this particular case, Professor Sugiyama is related to this company and making this software and maybe that's why he chose it; but there are many other softwares available which demonstrate a triangle and other stuff as well.

Becker: I have a couple comments. I think that the important thing here is the context in which the software is being used. We're talking about a lesson in mathematics that has specific objectives. The teacher knows what he or she wants to accomplish in the lesson. And the software has a role to play in achieving the objectives by the end of that class period or whatever the instructional period of time may be. That relates to Professor Fey's question about classroom management, which I think is exceedingly important. Also, as Jon Choate mentioned, maybe he would want to see this rectangle rolling around a circle. Well, that may be a very important question and that may be very important to investigate, but it also might be very important not to do it right now. Maybe that would be addressed in slightly different circumstances or at a different time.

Miwa: Well, let's think back to Professor Sugiyama's presentation during the whole class presentation and then the computer is used to supplement that. On the other hand, you can make the computer a priority and then in order to utilize the computer you may be able to change your teaching method or strategy. But maybe I disagree with both of them and feel there should be new development of how to coordinate the two opposing strategies. I would like to know Professor Sugiyama's opinion about this, though it is not in his presentation.

Sugiyama: The merit of the whole-class teaching is two-fold: (1) it is very effective to teach many students all at one time, and (2) the students can share their ideas and they can find out what they couldn't find out by themselves. And that's the reason why the whole-class situation is very popular in Japan. Well, now there is the individualized approach to the teaching and learning which is also very important, and that's where the computer is expected to come in and to supplement that idea. But, of course, there is a possibility of developing another way of teaching, but right now I am not sure of what it is.

Dugdale: Do you want to follow up, Professor Miwa?

Miwa: Well, maybe some other time I would like to get the opinions from the American side on how and what they think about this. For certainly, by introducing computer technology the teaching pattern may change and there may be great potential, and I would like to know other opinions as well.

Nohda: This afternoon in my presentation I am going to talk about the 20-30-70 problem and why the whole-class teaching is the major teaching technique in Japan. However, because of this we also have problems. For instance, in the upper grades of elementary school there is already twenty to thirty percent of the students who drop out or fail. In the case of the junior high school, it is almost fifty percent. When it comes to high school, seventy percent fail. And, therefore, while they see this problem with the whole-class teaching method, the introduction of computer technology into teaching may change or improve the situation.

Dugdale: One moment. For clarification of what has just been discussed, could Professor Miwa speak?

Miwa: Even though Professor Nohda used the word "drop out," it doesn't mean to drop out in the English sense. This was discussed at the last Seminar in 1986 also; at that time the expression 7-5-3 was introduced which means that, by the end of the elementary school level, 70% of the students understood the math or what they were learning. That means 30% did not quite understand. And at the end of junior high school, 50% had good understanding of the math. And at the end of the high school, 30%. That's how the 7-5-3 came about. So, the 7-5-3 doesn't mean that they have failed and checked out of school or even have failed the course. They just didn't understand everything.

Dugdale: I'm sorry we are at the end of our time. Thank you again to Professor Sugiyama, to Professor Kaji, and to Mr. Shimizu.

End of Discussion

INFUSING THE K-12 CURRICULUM WITH GRAPHING AND PROBLEM SOLVING

Frank Demana
Ohio State University

Bert Waits
Ohio State University

Hal Schoen
University of Iowa

The graphical and numerical representation of functions is a milestone in the history of mathematics that paved the way for such important developments as calculus, a topic receiving increased attention in recent mathematics curriculum recommendations (e.g., *Curriculum and Evaluation Standards for School Mathematics*, [NCTM], 1989), and an area in which curriculum and instruction could soon be revolutionized by the use of ever-improving graphing utilities--computer graphing software and graphing calculators. Graphing utilities have started to change the role of graphing in the school curriculum and in mathematics itself and corresponding changes in mathematical content from pre-algebra through calculus. Graphs of both functions and relations can now be obtained quickly with the use of technology such as the TI-81 graphing calculator used in this paper. Graphing can be used to infuse the curriculum with problem solving and to establish function as a major theme of the curriculum.

Establishing Functions Numerically and Graphically

Graphing should be started early in the mathematics curriculum and should be used to solve problems graphically. Significance and meaning can be established about graphing by using graphs to represent problem situations.

Prior to making graphs, tables can be constructed and used to explore and investigate problem situations numerically. The tables can be used to help students develop understanding about arithmetic processes, to foreshadow the study of algebra, and to solve problems numerically. Later the tables will be used to make graphs. This approach is illustrated in the following example.

Tables

Example 1. Shears Department Store is having a 20%-off sale.

(a) Complete the following table:

Original Price ($)	Sale Price ($)
10	
20	
30	
40	

Table 1. Sale Problem in Tabular Form

(b) What is the original price of a coat whose sale price is $30.40?

There are many benefits to using problems as a vehicle to develop understanding about mathematical concepts and to using a numerical approach to analyze problems (e.g., Demana & Leitzel, 1988). Using problems to introduce mathematical ideas helps students learn to value mathematics–an important goal of the NCTM *Standards*. This approach helps integrate problem solving into the study of mathematics and establishes problem solving as a major focus of mathematics. Exploring problems numerically builds understanding about problem situations that can be exploited when these problems are returned to in algebra. With the aid of calculators, students can do numerous computations in a given problem situation quickly. Repeated calculator-based computation in a given problem situation provides the necessary experience that allows students to establish understanding about arithmetic processes.

Additionally, the frequent use of tables helps establish function as a major theme of mathematics. Each table constructed gives students a concrete example of a function presented numerically. These tables can be used to construct graphs to represent functions geometrically and can be used to write expressions that represent functions algebraically (eg., see Comstock & Demana, 1987).

Problem Solving Numerically

Students can use the completed lines in Table 1 of Example 1 to get started on the solution to (b) of the example. The completed lines of Table 4 can be used to estimate that the answer to (b) is between $30 and $40 and a little closer to $40. Then, a guess-and-check approach can be used to determine that $38 is the answer to (b). Besides developing understanding about arithmetic processes, this rich arithmetic activity gives meaning to and understanding about finding solutions to such problems. Students can be guided to see that solving this problem numerically amounts to finding a line of the table with an entry in the second column equal to 30.40. This activity can be exploited later when students return to such problems and solve them graphically and algebraically.

The Role of Paper, Pencil, and Point Plotting

Much research is needed about how much point plotting by hand and with the aid of a calculator is necessary before students are turned loose with modern computer-based graphing tools such as a graphing calculator. Considerable point plotting is certainly necessary in the early grades, but if used exclusively, it may well interfere with students' understanding of the continuous nature of most graphs.

The Impact of the Graphing Calculator on the Curriculum

A major goal of the Grades 9 through 12 mathematics curriculum is for students to achieve in-depth understanding about important classes of functions. Prior to the study of calculus, this understanding would need to come from exploring numerous graphs quickly with the aid of technology. Teachers can guide this exploration so that students will actually conjecture statements that are true. In this way, students feel ownership in the mathematics. Teachers need to provide a few pitfalls to be sure that students use care when making conjectures (Demana & Waits, 1988a; Dion, 1990). However, this is not to say that students should be inundated with pathological examples.

A *complete graph* of a function $y = f(x)$ is a graph that shows all its important behavior (Demana & Waits, 1990b). What constitutes the important behavior of a function depends on where the student is in the mathematics curriculum. Important behavior of a function includes its y-intercept, zeros, relative extrema, and end behavior. In algebra and precalculus, teachers will have to tell students that a cubic polynomial function has zero or two relative extrema. And then students can find the coordinates of such extrema using graphical methods. In calculus, we would expect students to know *why* there are always exactly zero or two relative extrema for a cubic polynomial. Finding points of inflection would also probably be reserved for calculus.

The *end behavior* of $y = f(x)$ is its behavior for x large in absolute value. For example, the end behavior of $y = ax^2 + bx + c$ $(a \neq 0)$ is $f(x) \to \infty$ as $|x| \to \infty$ if $a > 0$, and $f(x) \to -\infty$ as $|x| \to \infty$ if $a < 0$. Notice that the end behavior of $y = ax^2 + bx + c$ and $y = ax^2$ are the same because the values of $ax^2 + bx + c$ are dominated by the values of ax^2. In other words, $y = ax^2$ gives a model of the end behavior of $y = ax^2 + bx + c$. We say that $y = ax^2$ is an *end behavior model* of $y = ax^2 + bx + c$. More precisely, g $(\neq 0)$ is an end behavior model of f if and only if $f/g \to 1$ as $|x| \to \infty$. The end behavior model concept is also very powerful visually: for example, the graphs of $y = -2x^2$ and $y = -2x^2 + 5x - 7$ in the viewing window [-1,000, 1,000] by [-1,000,000, 500,000], that is, in the rectangular region of the place given by $-1{,}000 \leq x \leq 1{,}000$ and $-1{,}000{,}000 \leq y \leq 500{,}000$, are *visually identical.* This is convincing evidence that the behavior of $y = -2x^2 + 5x - 7$ is essentially the behavior of the simpler function $y = -2x^2$ for large values of $|x|$.

We would expect that prior to the study of calculus students will, through the power of visualization, have in-depth understanding about polynomial functions, radical functions, rational functions, exponential and logarithmic functions, trigonometric functions, conics (relations), and polar and parametric equations. The class of rational functions, for example, provides important foreshadowing activity for the later study of calculus as illustrated in Example 2.

Example 2. Determine a complete graph of $f(x) = \dfrac{x^4 - x^3 - 6x^2 + 5}{x^2 - x - 6}$

Figure 1 gives the graph of f in the viewing rectangle [-7, 7] by [-10, 30]. Notice that this graph illustrates the end behavior of f because it suggests that $f(x) \to \infty$ as $|x| \to \infty$. Some beginning students will need more detail near $x = -2$ and $x = 3$ to be sure that f has vertical asymptotes at $x = -2$ and $x = 3$. The graph of f in Figure 2 gives strong evidence that f has three local extrema in $-2 < x < 3$. Notice that Figure 1 shows f has two other relative extrema outside this interval for a total of five extrema. We can zoom-in on the graph of f in $-2 < x < -1$ to see that the graph crosses the x-axis twice in this interval. Thus, f has four real zeros. Prior to the study of calculus, teachers will need to guide students to discover that the graphs in Figures 1 and 2, together with more detail around $x = -2$ and $x = 3$, if needed, constitute a complete graph of f. This example illustrates that more than one graph is often necessary to show all of the important behavior of a function.

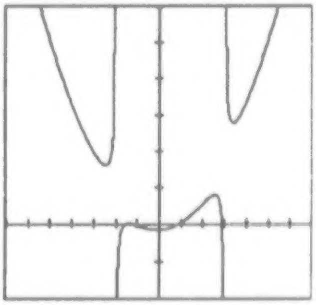


Figure 1. The graph of

$f(x) = \dfrac{x^4 - x^3 - 6x^2 + 5}{x^2 - x - 6}$

in [-7, 7] by [-10, 30].

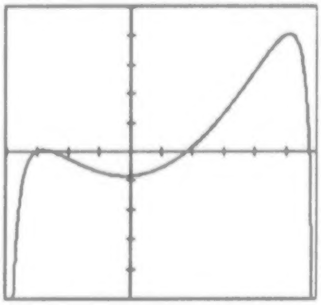


Figure 2. The graph of

$f(x) = \dfrac{x^4 - x^3 - 6x^2 + 5}{x^2 - x - 6}$

in [-2, 3] by [-5, 5].

Students can use algebraic manipulation to help understand the graph of f. Using long division and factoring, f can be rewritten as follows:

$$f(x) = \frac{x^4 - x^3 - 6x^2 + 5}{x2 - x - 6} = x^2 + \frac{5}{(x+2)(x-3)}.$$

In this form, students become convinced that the graph of f looks like the graph of $y = x^2$ away from $x = -2$ and $x = 3$ and "blows up" near $x = -2$ and $x = 3$, as suggested by Figure 1. This form also suggests that $y = x^2$ is an end behavior model of f, because it is easy to see that $f/x^2 \rightarrow 1$ as $|x| \rightarrow \infty$. An end behavior model of f can also be discovered graphically using *zoom out*, that is, viewing the graph of f in large viewing rectangles. The graphs of f and $y = x^2$ will appear to be nearly identical in the $[-100, 100]$ by $[-10{,}000, 10{,}000]$ viewing rectangle.

In calculus, students can compute the derivative of f to see that it is also a rational function with the polynomial of degree 5 as the numerator. Since a polynomial of degree 5 has at most 5 real zeros, the calculus student can now conclude that, based on the graph of f in Figure 1, f has exactly five relative extrema. In calculus, conjectures about the behavior of functions made in precalculus can be proven. This is a modern role of calculus.

Results from Recent Research

Rich (1990) found that students who are taught precalculus using a graphing calculator better understand the connections between an algebraic representation and its graph and that they view graphs more globally, in that they understand the importance of a function's domain, the intervals where it increases and decreases, its asymptotic behavior, and its end behavior. Browning (1988) found that high school precalculus students who used graphing calculators for one year exhibited a significantly increased ability to deal with graphing at the more advanced Van Hiele levels of analysis and ordering. Farrell (1989) also observed that precalculus students who were taught the use of graphing calculators demonstrated greater facility with higher-order thinking skills than traditional students. Further, Dunham (1990) observed that in college algebra classes requiring graphing calculators, gender-related differences in performance on graphing items were eliminated, while pretest performance on graphing items indicated that females performed at a lower level than males.

Because graphs are easily obtained, it is reasonable to emphasize that finding all the real solutions of $f(x) = 0$ is the same as finding all the x-intercepts of the graph of f.

Limiting ourselves to algebraic techniques seriously restricts the types of equations students can solve. In the conventional curriculum, students solve linear equations, quadratic equations, easily factored *contrived* higher-order polynomial equations, and other contrived equations whose form is special. However, using computer graphing, students can easily solve very complex equations, *even equations that do not admit an algebraic solution.* The graphing method is illustrated in the following example. We could easily handle more complicated examples the same way.

Example 3. Solve $x^3 + 2x = 1$.

It can be shown that the graph of $f(x) = x^3 + 2x - 1$ in Figure 3 is complete. Thus, the equation has one real solution. Because the scale marks are one unit apart in Figure 3, the one real solution lies between 0 and 1 and is about 0.5. We can create a decreasing nested sequence of viewing rectangles that converge to the x-intercept and that allows us to determine the x-intercept with accuracy within the limits of the precision of the machine in use.

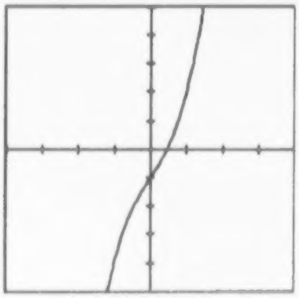


Figure 3. **The graph of $f(x) = x^3 + 2x - 1$ in [-4, 4] by [-5, 5].**

The scale marks on the horizontal axis in Figure 4 are 0.1 apart. This allows us to read the solution to the equation as 0.45 with error of at most 0.1, the distance between the horizontal scale marks. Similarly, we can read the solution as 0.454 with error of at most 0.01 from Figure 5, and as 0.4534 with error of at most 0.001 from Figure 6.

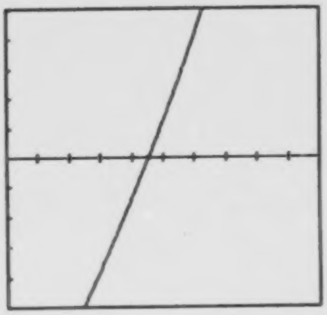


Figure 4.
The graph of
$f(x) = x^3 + 2x - 1$
in [0, 1] by
[-0.5, 0.5].

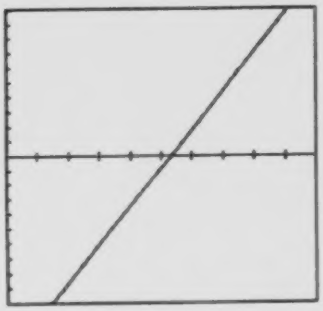


Figure 5.
The graph of
$f(x) = x^3 + 2x - 1$
in [0.4, 0.5]
by [-0.1, 0.1].

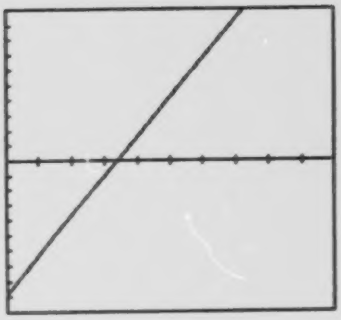


Figure 6.
The graph of
$f(x) = x^3 + 2x - 1$
in [0.45, 0.46]
by [-0.01, 0.01].

The process illustrated by Figures 4-6 is called *zoom-in*. Modern graphing calculators, such as the TI-81, have automatic zoom-in as a feature. However, the careful selection of viewing rectangles suggested by Figures 4-6 is a good beginning activity for students using graphing utilities. This process can be continued until we determine the solution of the equation with desired accuracy within the limits of machine precision.

The situation for solving inequalities is very similar to solving equations. The algebraic techniques are limited, causing a corresponding restriction on the types of inequalities that can be solved. Again, solving inequalities graphically allows a wider variety of complicated inequalities to be solved: for example, to solve $x^3 + 2x < 1$ (or, $x^3 + 2x - 1 < 0$), we need to determine the values of x for which the graph of $f(x) = x^3 + 2x - 1$ lies *below* the x-axis. Figure 6 permits us to read the solution to the inequality as $(-\infty, 0.4534)$ with an error of at most 0.001.

Technology allows algebra students to solve equations such as $500 = x(20 - 2x)(30 - 2x)$ to find the side length of the squares that must be cut from each corner of a 20 cm by 30 cm piece of cardboard in order to form a box with no top that has volume 500 cubic cm. These are equations that we could never dream of asking students in school mathematics to solve without graphing calculators. Solutions are now possible because of the graphical techniques available with today's technology.

The Modern Role of Algebraic Manipulation

Much of the present K - 12 curriculum consists of mindless arithmetic drill and algebraic manipulative practice. The *Standards* call for this type of activity to be substantially reduced and replaced by activities designed to foster higher-order thinking skills. This is not to say that arithmetic and algebraic manipulation should be completely removed from the curriculum. In light of technology, we now need to ask ourselves why we do these things and we must try to give students reasons (Demana & Waits, 1988b). Students will view these skills as important if they grow out of applications or out of situations where other mathematical understanding is obtained. In other words, these rote activities should not be the focus of the majority of lessons. Rather, they need to be used in more interesting activities.

Graphing receives very little attention in the current curriculum. Graphing utilities are and will continue to have a profound effect on the upper curriculum. Students need to learn how to use graphing as a problem-solving tool. The curriculum needs to be infused with realistic problems to help all students learn to value mathematics. The use of graphing calculators will help make the vision of the new NCTM *Standards* a reality.

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Discussion of Professor Demana's paper

Hashimoto: Thank you very much for a very nice presentation. We shall now proceed with the discussion. Are there any questions and comments?

Sawada: Yes, I have two questions concerning the use of technology: (1) If the student can do the characterization but when it comes to graphing they tend to get lousy and cannot, for example, distinguish between a straight line and a curved line, how do you treat these things? (2) Are the symbols used in the regular math and math with technology somewhat different and how do you treat them?

Demana: In first case the reason I use that rational function is because our students prefer to draw rational functions by hand. They find them very hard to do on the machine and so our students do long division, divide the denominator into the numerator, get the quotient which is this end behavior idea. They actually draw that and then erase around, as they say "erase around the bad points," and complete the graph that way so that, in fact, we get more ordinary by hand drawing than we ever did before. We take building blocks. We start with x^2, we make sure they can very carefully shift up and down, left and right, stretch, and we go through building blocks to do that with every polynomial one over x, e to the x, and log x, so that when we get to the transendental functions they are, in fact, done. So, that's question 1. The second question was about confusing linear and curve. In fact, when I went into a fourth year course which meant they had algebra I, algebra II, and geometry, they, in fact, could not graph linear and quadratic functions. And that's why we had to really start over with that. Now our students understand classes of functions. When they look at a polynomial of degree 3, they know what the possibilities are. When they look at a polynomial of degree 4, they know what the possibilities are. They even know how to graph, as I mentioned, these rational functions...get a complete graph. Let me illustrate that with a hand drawing so you can see what I mean. If they did the division so that it became say $f(x) = x^2 + 1$ over x–2, here's what our kids do. Well, I'm going to cheat a little because I don't like to get this stuff on my hand, but they actually draw x^2 . They use pencil and then they erase. But I'm going to erase at the same time. Okay? And then they come in and fix it up...like that. Now, our precalculus students think about rational functions as polynomials with a few bad places and our calculus kids never had that kind of understanding before.

Sawada: Well, when you draw the curve manually you can draw a nice smooth curve, but when it appears on the calculator screen, the line tends to be quite rough and broken. How do you explain to students about that? Also, the symbolic expressions you use may not be exactly the same as those you use manually; for example, x^2 looks like x^2 and you have to use different symbols. So, what's the advantage or the difference? Also are you going to do the same thing with a lower level or lower grade students?

Demana: The answer is yes. We're doing it in algebra. We're actually doing some stuff in middle school, graphing the very first example (demonstration on OHP and screen). We talk about the roughness. We talk to students about how the computer (that's what this calculator really is) is actually graphing, There is a mode on here where I can forget about the smoothing function and see what it's using. So we can talk about what the machine is really doing. Let me take some easy examples here. I'll get rid of the hard one. I'm just going to press in x^2. Incidentally, we can get two exponents to your right and after that it's carrot like a computer, and it does understand juxtaposition-writing next to each other. 2x makes sense, you don't need 2*x. But now when I go ahead and graph this, yes, it's a little rough, there is a mode in here which allows me to turn off this feature and I can see what the machine is doing. It's taking those points and I can talk about it's trying to draw a smooth curve. I can even tell it, don't use as many. I can go in here and I can say, don't use every pixtel, use every other one, for example, and I'll get fewer. And when I complete that it'll look rougher than when I fill it in. So we use it as an opportunity to actually talk about how it's doing it so that, in fact, you don't get misled. See, now that one's a little rougher than the other one and I can make it rougher yet depending on what I choose as the resolution. So you do have to be careful, and we just make kids understand what the graph really looks like. This is only a tool in helping you to understand.

Sawada: As a math educator are you requesting the manufacturing companies to make a finer machine so that the it can draw a much smoother graph or curve?

Demana: Not really. I want it to have more features ultimately. To do more things. But I don't want it to be so smart that the student has nothing to do. I use the pictures as a way of creating student questions, promote discussion in the classroom and then we talk about the mathematics that's always been there. We use these graphs for example so that when we come back to it in calculus, we can talk about how do you find where it has relative extrema. Well, we know the derivative should be zero and we can actually pop graphs up at the same time. I'm not concerned with it being correct completely. I get a lot of mileage if it's not always correct, or as smooth as possible.

Sawada: I just wanted you to know that one calculator manufacturing company in Japan manufactures a calculator similar to yours; while it was not used very widely, the math teachers told them it was not accurate enough (the graphs were not smooth and precise) and they would not use it. But when this same calculator was available in the U.S., teachers thought well of it.

Demana: Yes. We, in fact, used Casios for a full year and it was the teachers and students using Casios that prompted the features of this particular machine. This has built into it the things the teachers and students didn't like about the Casio, or built out of the old part of it.

Morimoto: Japanese math teachers still feel this resolution is too rough, it's not exact enough yet.

Are you satisfied with this level of resolution or, if it is financially possible, are you thinking of switching to the more refined resolution calculator.

Demana: I would take more resolution, but with the understanding that the price has to be low. These cost less than $80 in the U.S. We are able to put them in every kid's hand. We tell parents, for example, they cost less than a pair of good tennis shoes. And if you're going to give me a $1000 machine with lots of power in my hand, I'm not going to be able to use it. So I'll take the best I can get for the money, but the money is a driver. No matter what you use, no machine can be exact. You build it as good as you want and then I'll ask you to graph sine nx and I'll make n big enough that it will scuz up your machine. You can't make it do that, I mean, you can't ask machines to do something that's impossible to do. But you can ask students to understand what resolution means and that's important.

Dugdale: Similar to the question of resolution and smoothness, I am not so bothered by the lack of resolution, but I am more concerned by the behavior of the graph around discontinuities. I am used to software that recognizes a discontinuity, does not plot the asymptote as part of the graph, and shows the appropriate behavior of the graph around the asymptote. That's different from what Frank is showing, but Frank is using the lack of accuracy around a discontinuity as an opportunity for students to do some thinking, and it sounds to me like that's one of Frank's examples of not wanting the software so smart that the student has nothing to do. I would like your comment on that, and correct me, Frank, if I've misinterpreted you. I am also interested in the Japanese perspective on whether it is the general, overall smoothness of the graph that you find a problem or whether it is the critical points in a graph where it does not plot correctly because of the resolution.

Hashimoto: Does anyone in Japanese delegation have a comment? Professor Miwa?

Miwa: This discussion has come up because Sawada commented that the Casio calculator manufacturer produced a calculator with low resolution, below Japanese math teachers' standards. If that is clarified, probably that would answer the question. So, perhaps Sawada might respond to the question.

Sawada: First of all, certainly there is the problem of the continuity of the curve, the smoothness and also the relationship between the curved line and an asymptote. This is a very big problem and it's been asked on examinations so that, in the Japanese high school, they're dealing a lot with this. And that's why this smoothness came into question. Now as far as the curriculum and calculator are concerned, the present curriculum is made by Monbusho (the Ministry of Education) and does not include calculator use and, therefore, there's no room to use them. No textbooks include them either. I'm not talking about the new syllabus which will be in effect in 1994, but the present one that is being used. So, in Japan, of course, the graphing calculator is available but still teachers are not using them enough. For example, most calculators, unlike what was shown today, don't have the zooming function or show the intersection of two lines very clearly and you cannot really see it. And those are the reasons why Japanese math teachers do not adopt calculators today for their teaching.

Hashimoto: I hope the U.S. delegation can understand the Japanese situation. Are there other comments? Mr. Iida.

Iida: In the American system of math education which has the priority - the teacher's explanation, the lesson/lecture or student participation in the learning activity? Also, is there any opportunity for the students to express their own opinions or present their own ideas? In this case, do teachers fully understand what the students are thinking? For instance, in Japan and in general, teachers grasp students' way(s) of thinking very clearly because they let them introduce their own ideas or opinions in class; but here in the States, if you let the students talk about their own opinions, do teachers fully understand what they are talking about?

Demana: I would say that is part of the reason for the strong use of technology in our system, to turn our classrooms away from a demo, that is the teacher presenting... students dutifully sitting and listening, without doing anything. Technology does get the student involved. Now it's been hard for teachers because they have been used to having a classroom under their control, and feel threatened when asked questions they can't answer. And you saw yesterday questions that can come alive that the best mathematician around is not able to answer from these visual experiences. Now, once our teachers get past that fear, and are partners with the students, a guider and explorer on an adventure, then our students are really excited about it because they find out mathematics is not dead. There are lots of interesting questions. At Ohio State University we have been training teachers in one-week institutes since 1988 in the use of this calculator and that's been one of the key issues, but we have unbelievable interest in this in our country, as I said. We will have, by the end of the summer, put through 1500 teachers in one-week institutes around the country in classroom teaching this way and they report back that although it's hard the first year, it is a good way to operate with shared experience when they get in the classroom. They're working together, truly working together.

Uetake: With this calculator made by Texas Instruments, can you use parametric equations and polar coordinates?

Demana: Do you want me to do it now? Sure.

Demana: Okay. I'll do them both at once. Is that what you want (demonstration)? Or do you just want an answer to a question?

Uetake: Yes, please show us.

Demana: How about short 0 mathematics, how's that (demonstration)? Here's short 0 mathematics. I'm going to put it from function to parametric and now I have 3, ahhh. I'm going to leave what I've got in there cause that's what I was going to do except I'll come down here and change this to sine. ooops. that isn't what I want. The first one parameterizes the unit circle. It's going to have center at –10, the second one parameterizes a function sin x. okay. Now I'm going to fix the range where you can see what that is, that's 0 to 2π. The t-step here you do and control the resolution in parametrics, so you can close up some of those other issues you want, but still you're limited by the number of pixels on the screen. I only need here about negative, oh, let's make it 4 and let me only go from negative 2 to 2 and let me do one other thing. Let me graph them both at the same time. You may not like that - that thing doesn't quite look like a circle but I can fix that quickly for you but if you look at this, this is the sine function unwrapped, this is the wrapping function. If I use the trace and go over here I'm on the unit circle where the y coordinate is the sine and I can jump to the others to see what's happening. As I go around I can see the way in which the sine has been graphed. So I've unwrapped the sine function but the nice thing is you don't always look at these others. Here's an unwrapping of the tangent function. Now, what Sharon asked a minute ago which I didn't get to (want to do a square), I can square this up so that it makes a circle look like a circle. Sharon asked do I care about those singularities. This is where I want to go in and switch to dot mode. I don't want it too smart. I get some mileage out of kids zooming in on singularities. There's the tangent. I can unwrap all the functions. That shows you how to do this but, more than that, we actually can simulate motion. We can do real problems. One of the neatest problems that our students get into is hitting a baseball and deciding whether it's going to be a homerun or not. And you can even put a drag factor on that for the wind, you don't have to worry about gravity only, you can put air resistance in. You can make it not contrived, you can make it visual. And this is the old thing you've always tried to teach, that's the unwrapping function modernized.

Uetake: What about polar coordinates?

Demana: Well, polar, it doesn't do polar directly, you got to convert. So what you have to do here, let me get rid of these things, let's say you want to do 7 sine of 3 T, 7 sin 3 T. You have to think about x being r. This is r = 7 sin 3 T. Then x is r cosine T so we use it as an opportunity to make that connection. So now I get 7 sin 3 t sin t. I've now parameterized it and I'll just do zoom 6 so you'll see it and there is the familiar one. But now in an exploratory way, I can go back here and once it's done I can go back here and change the 3 to a 4, for example. Ahhhha, this is what the students did to our teachers. And this is where you have to worry and be ready. The kids, once they could play with this, are not stuck in the same rut we are and they'll do it.

Wilson: I will enter the end of that as my comment.

Demana: Well, I didn't give up. I put it in. That sin 2.5, 7 sin 2.5 t and you don't have a complete graph and the kids ask that question, not the teacher. Teachers like me are so used to sin theta when n is even and odd you never thought beyond because of what we were stuck in before. This is a natural question for a kid and they found out that, in fact, the ten leaves there overlap and they can find the period, but the real lesson for our teachers was you don't know what the kids are going to ask and for sure you may not be able to answer it. But that's, I claim, that's a healthy experience.

Hashimoto: Okay, thank you very much again, Professor Demana. I'd like to close this session now, thanking you very much for you cooperation.

End of Discussion

STUDY OF PROBLEM SOLVING WITH CABRI-GEOMETRY IN SECONDARY SCHOOL MATHEMATICS

Nobuhiko Nohda
Institute of Education, University of Tsukuba
Tsukuba, Japan

Background

In Japanese schools, mathematics is considered a key subject. In fact, mathematics is compulsory in elementary school (six years) and lower secondary school (three years), called junior high school. In upper secondary school (three years for full-time course), called senior high school, all first-year students have to take Mathematics I; but in the subsequent years mathematics is optional. The objectives and content of the mathematics curriculum are determined in the Course of Study. The Japanese Course of Study is set by the Ministry of Education and has become the standard syllabus. Textbooks used in class are compiled by commercial publishers according to the Course of Study and have to be examined and approved by the Ministry of Education before being used in schools. Details and teaching methods are left to classroom teachers.

Computers are introduced in the Japanese Course of Study in junior and senior high school, which should be efficiently utilized as the occasion arises. In particular, this matter needs to be considered in the instruction by experiment, observation and etc. on problem solving. The availability of computers in mathematics classrooms provides unique opportunities not only to take advantage of the motivational effects of real-life problems, but also to develop useful methods for attacking basic problems within the mainstream of secondary school mathematics.

In 1989, the Japanese Ministry of Education recommended the use of computers and hand calculators in the latest version of the educational curriculum. Computers and hand calculators are emphasized more in this version than in the former one. This curriculum was designed to help students have a good sense of mathematical thinking and with a goal that students should become good problem solvers. In this new curriculum, the soroban (abacus), hand calculator and computer are given more attention in teaching mathematics in class. Opportunities are also provided for Japanese students to consider appropriate situations for conjecturing and checking the results of problem solving. All of the recommendations above are to be implemented in elementary to secondary school mathematics curricula beginning in 1992 for elementary, 1993 for junior and 1994 for senior high school.

Computer environments are ideal tools to support the implementation of the junior and senior high school curricula, especially in mathematics. For example Tall's (1985, 1986) research is based on a cognitive approach to the curriculum. It uses software that is especially designed to enable the user to manipulate generic examples of a specific mathematical concept or related system of concepts and thus to grasp a gestalt for a whole concept at an intuitive level. The learner is directed through a suitable sequence of activities with examples and non examples towards the generic properties of the concept. This dynamic process helps him or her to construct version of the concept by graphic calculus.

Tall (1986) and Schwarz (1989) have used experimental and control classes and shown the experimental curricula to be clearly superior to the standard curricula in which they were designed to do. It also appears that visualization plays an important role in this development and that open-ended learning environments are tools that are well suited for presenting mathematical topics in a manner that stresses objects and processes while using visual and analytic descriptions in parallel (Balacheff, 1990).

Problem solving with computer

We will talk about the process of problem solving. There are many assertions about the process of problem solving. For instance, one is G. Polya's (1957) four-phase description of problem solving activity. The four-phase involves understanding, planning, carrying out the plan and looking back. On the other hand, the conjecturing of problem solving processes and results are some kinds of higher order thinking. These viewpoints lead to the usefulness of conjecturing in understanding a problem and in making a plan, and in evaluating decisions made and outcomes of executed plans. If we take these viewpoints, then problem solving processes are a vital context in which to learn and appreciate mathematical thinking.

We would like to make in details, a comparison using computer with paper-pencil computation in the situations of mathematical problem solving. There are some thinkings about the tools of computations with computer in its relation with paper-pencil. One thinking is that, computer is one of computation, the same as paper-pencil computation. Another thinking is that, computer serve as the compensate for calculator and paper-pencil computation. Furthermore, we are thinking that the conjecturing and demonstration with computer are some kinds of problem solving processes. It is important to study the value of the use of computer in mathematical problem solving. It is important for students to recognize that computer is useful for understanding a problem, making a plan, checking the way of solving of the problem, decision making and outcomes of executed plans.

Cabri-Geometre

We will introduce the software of Cabri-Geometre which Dr. Jean-Marie Laborde mentioned (1990). Cabri-Geometre (Cabri stands for CAhier de BRouillon Interactif/ Interactive Notebook, Baulac, Bellemain, Laborde, 1988) consists of a package for constructing geometrical figures that can be used for teaching and learning geometry. Cabri Geometry deals with points, lines, circles and their relations and allows the user to realize geometrical constructions. Due to its internal representation, the software offers the possibility to move around any of the basic points of a figure by direct interaction with the mouse. The user continuously sees the figure redrawn in real time keeping all its initial properties. This basic feature of Cabri makes the user consider a figure not as a static drawing but as a set of objects linked by geometrical relationships. The user can manipulate figures freely.

Cabri-Geometre is a microworld. Even if there is no standard definition of what a microworld is, most authors would probably agree on the following formulations: A microworld is an (often computer based) environment

- which provides a set of primitives (objects and activities) that can be combined in order to produce intended effects (computational, graphical, ...),
- which offers a variety of different ways to obtain an intended effect,
- which embodies an abstract domain described in a model,
- which is open-ended since it can be used to produce a variety of different, effects that are partially related.
- A last criterion could probably be added: an implemented microworld should offer the possibility of direct manipulation of the objects.

As in any microworld, Cabri-Geometre encourages the learner to explore the environment, here the world of Euclidian Geometry. Because it is easier to use (partially insured by direct manipulation), the learner can get an idea of what geometry is. Figure 1 illustrates an interesting situation. Here we have a triangle ABC and its reflected image, but if we consider the

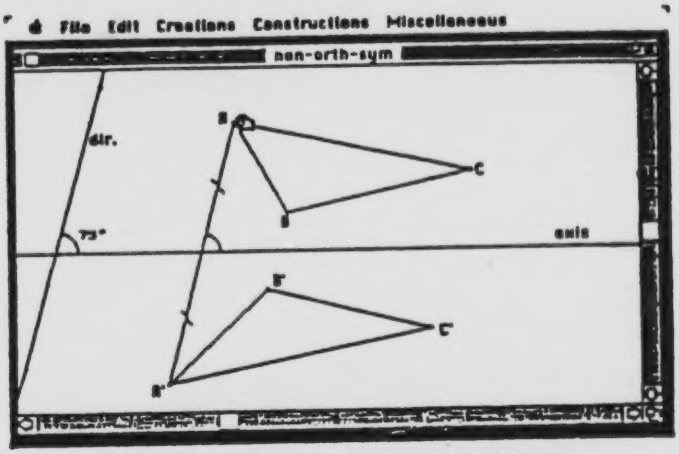


Figure 1: A typical screen of Cabri-géomètre on Macintosh
(observing the reflected image of triangle ABC when moving A in a non orthogonal symmetry)

Figure 1

circumscribed circles to ABC and to A'B'C', they do not meet at the symmetry axis. The following could be used, for instance, as a starting point of a teaching sequence in which conceptions of students about geometrical transformations would evolve: a transformation does not necessarily preserve the nature of a geometrical object (as is the case for line-segments, straight lines, and triangles, in the case of reflection). Here constructing the image of a circle requires considering the circle as a set of points and constructing the image of each of these points. Thanks to the menu-item locus of points" shown in the Figure 2, this is easily done in Cabri-Geometre.

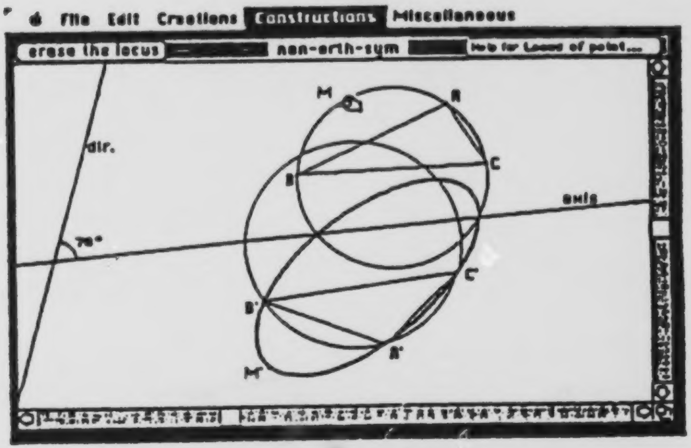


Figure 2: Using Cabri-géomètre
(the circle going through A'B'C' is different from the image of the circle through ABC)

Figure 2

Mathematical Problem Solving with Cabri-Geometre

This research had been studied in the France and Japan Collaborative Research three years ago. Members of France and Japan Collaborative Research (University of Grenoble and University of Tsukuba) are as follows:

French members:	N. Baracheff, J-M. Laborde, E. Gallou-Dumiel, M. Picq.
Japanese members:	N. Nohda, K. Nakayama, Y. Higashibara, K. Kakihana, K. Harada, M. Miyazaki, K. Shimizu.

This research was divided into two parts of studies on mathematical problem solving with Cabri-Geometre in Japan:

(A) The effect of Cabri-Geometre on studying plane geometry in junior high school: Ms. Kakihana, Prof. Nakayama, Prof. Higashibara, Dr. Shimizu and Prof. Nohda.

(B) Student's activities of the problem solving with Cabri-Geometre on finding the number of diagonals in a polygon: Mr. Miyazaki, Prof. Nakayama, Prof. Higashibara, Dr. Shimizu, Ms. Kakihana and Prof. Nohda.

(A) The effect of Cabri-Geometre on studying plane geometry in junior high school by Ms. Kakihana.

(1) Aim of the study:

This study aims to investigate the effects of Cabri-Geometre" when solving an open- ended problem of plane geometry in junior high school.

(2) Procedure of the study:

By comparing an experimental group with the software of Cabri-Geometre and a control group with paper-pencil, the effects of problem solving on the following aspects were examined:

- (a) The possibility of making conjectures in the open-ended problem
- (b) The changing of the image toward a figure of plane geometry
- (c) The time of manipulations of Cabri-Geometre

(3) Subjects:

The following experiments were conducted at the junior high school attached to the University of Shimane in February 1991. The two following groups were used for the experimental and control classes:

(a) Experiment group: 33 students in the seventh grade
41 students in the eighth grade
(b) Control group: 40 students in the seventh grade

(4) Procedures:

- (a) The experimental group practiced with Cabri-Geometre for two hours before the experiment was done. After several days they solved the problem with Cabri-Geometre for one hour (cf. Figure 3). Each student used his/her own computer.
- (b) The control group solved the same problem without the computer.
- (c) The scores were calculated according to the scoring criterion (zero (no answer) to 5 (correct answer); see Table 1 in Appendix)) and statistics of each group were calculated.
- (d) The experimental group took pre and post questionnaires about the image of the figure of plane geometry by the SD (Semantic Differential) method.
- (e) Manipulations of the Cabri-Geometre by students were observed and the time to finish the construction of the figure was recorded.

(5) Experimental Problem

Here is triangle ABC. Point D is the middle point of side BC and Point E is the middle point of side AC. Point F is the midpoint of line segment BE and Point H is the midpoint of line segment EC. Then you can construct a quadrilateral DHEF. If Quadrilateral DHEF has the following shape of figure, what kinds of shape will the triangle ABC have?

(6) Procedures of Problem Solving

(a) If Quadrilateral DHEF is a parallelogram, then the triangle ABC can be any triangle.
(b) If Quadrilateral DHEF is a rectangle, then the triangle ABC is either an isosceles triangle or an equilateral triangle.
(c) If Quadrilateral DHEF is a rhombus, then the triangle ABC is a right triangle.
(d) If Quadrilateral DHEF is a square, then the triangle ABC is a right is isosceles triangle.

Shape of quadrilateral	Shape of triangle ABC
Parallelogram	
Rectangle	
Rhombus	
Square	
Others	

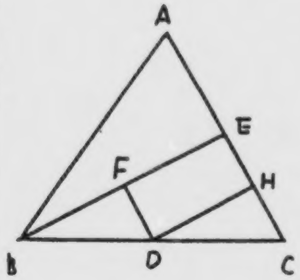


Figure 3. Problem and Procedures of Problem Solving

(7) Results and Discussion

(a) Making conjecture in the open-ended problem

(a-1) Comparing the significance of the mean scores of the difference between the experimental group and control group of Eighth grade (Table 2). Students of the control group scored better on the parallelogram and there was no significant statistical difference between the experiment group and control group on the rectangle. In the case of both the rhombus and square, the null hypotheses were rejected.

The reasons why students of the control group scored better on the parallelogram were because they were familiar with the parallelogram before learning it in plane geometry and they responded to only the first question.

Table 2 The mean scores of experimental and control groups.

	Ex. Group	Con. Group	t	df
parallelogram	3.32 (2.16)	4.66 (1.15)	-3.50	80 **
rectangle	2.88 (0.84)	2.70 (1.29)	0.71	80
rhombus	3.39 (1.18)	2.10 (1.55)	4.26	80 **
square	2.51 (1.45)	1.76 (1.45)	2.38	80 *

Note: Significance Level * : 5%; ** : 1%

(a-2) Comparing the significance of the mean scores of difference between the experimental group of eighth and seventh grade (Table 3), in all cases the null hypotheses were rejected.

Table 3 The mean scores of experimental Eighth and Seventh.

	Eight Grade	Seven Grade	t	df
parallelogram	3.32 (2.16)	2.45 (1.33)	2.00	72 *
rectangle	2.88 (0.84)	2.09 (1.68)	2.62	72 *
rhombus	3.39 (1.18)	1.48 (1.70)	5.68	72 **
square	2.51 (1.45)	1.42 (1.60)	3.06	72 **

Note: Significance Level * : 5%; ** : 1%

(a-3) Comparing the significance of the mean scores of difference between boys and girls included sum of total scores (Table 4). There was no difference between boys and girls. As far as we observed the lessons with Cabri-Geometre, lots of boys seemed to be manipulating more quickly and using the function of computer better than girls.

Table 4 The mean scores of boys and girls

	Boys	Girls	t	df
Second Grade	13.10 (3.92)	11.00 (3.12)	1.86	39
First Grade	9.00 (4.44)	6.06 (5.15)	1.75	31
Contr. Group	11.30 (3.69)	10.90 (3.29)	0.32	39

Note: Significance Level * : 5%; ** : 1%

(b) Changing the images of Plane geometry
After the lesson with Cabri-Geometre, students' images of plane geometry at seventh grade shifted from the feeling of simple, ordinary and old-fashioned to a better feeling of sophisticated, diverse and modern-fashioned. Eighth grade students shifted from the feeling of difficult, troublesome and old-fashioned to some better feeling of easy, simple and modern- fashioned, but from one feeling of rewardable to rewardless.
(These data are omitted)

(c) Time of manipulation with Cabri-Geometre

(c-1) Five students of Eighth grade finished the complete figure of the given problem in five minutes. They mastered quickly the operations of Cabri-Geometre. One of the seventh grade students finished the complete figure of the given problem in forty minutes. He was the slowest paced.

(c-2) During the practice time, almost all the students had mastered the manipulation of the basic operations with Cabri-Geometre in fifty minutes. Later on, they were able to construct a simple figure of another problem in plane geometry.

(B) Student's activities of the problem solving with Cabri-Geometre on the finding the number of diagonal in a polygon by Mr. Miyazaki.

(1) Aim of the study:

This study aims to discover the conditions which lead to conjectural activity for students, to study the relationship between the types of conjectures and mathematical content and environments with a computer, and finally to analyze the production of conjectures and problem solving strategies by students.

This scheme is taken from Fishbein's research on proof; its aim is to provoke the elicitation of students' points of view on proof, which will then be compared to their problem solving behavior. The questions are as follows:

- What is a counterexample?
- What is a proof of the solution of a mathematical problem?
- If a statement has been proven, does the verification of more cases provide more certainty?
- If a statement has been proven, and if a case is found such that the statement is not true for it, what is the meaning of such an event?

These interviews will be analyzed from a cultural point of view, looking for aspects in pupils' conceptions likely to prove the existence of differences in their school culture; that is, differences in the didactic contract, or in the perceptions of mathematics as a socialized knowledge in each country.

(2) Procedure of the study:

By comparing between an experimental group with the software of Cabri-Geometre and a control group with paper-pencil, the effects of problem solving on the following aspects were examined:

(a) What is a counterexample?
(b) What is a proof of the solution of a mathematical problem?
(c) If a statement has been proven, does the verification of more cases provide more certainty?
(d) If a statement has been proven, and if a case is found such that the statement is not true for it, what is the meaning of such an event?

(3) Subjects:

The following experiments were conducted at the public junior high school in Tsukuba City, Ibaraki Prefecture, in May 1991. The two following groups were used for the experimental and control classes:

(a) Experimental group:	six pairs (12 students in the seventh grade)
(b) Control group:	six pairs (12 students in the seventh grade)

(4) Procedures:

(a) The experiment group practiced with Cabri-Geometre for about one hour before the experiment was done. After several days they solved the problem with Cabri-Geometre for one hour. Two students used one computer.
(b) The control group solved the same problem without a computer.
Two students used one pencil to solve the problem.
(c) After students' problem solving, the twelve pairs were interviewed.

While each student was being interviewed, it was video-taped at the same time.

(5) Experimental Problem

Please write your way of thinking about how to find the number of diagonals of a polygon, and explain your reason why your way of thinking is correct to your friend in the class, when we have already known its vertices.

(6) Problem solving activities by both Students(SM) and (SH)

(a) Processes of problem solving with cooperative activities by SM & SH.

(a-1)The way to find the number of diagonals of a polygon.

The method: If the number of angles is a, we can find it by $(a^2-3a)+2$

The reason:

A: The number of angles

B: The number of diagonals

C: The minimum number of triangles produced by diagonals

D: The number of diagonals from one vertex

A	B	C	D
3	0	1	0
4	2	4	1
5	5	10	2
6	9	18	3
7	14	28	4
.	.	.	.
.	.	.	.

We first make this table. We find $B = C/2$

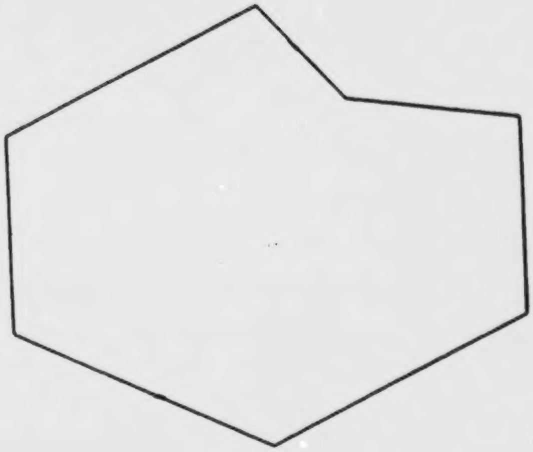
Furthermore, we find $D = A-3$

And , we find $AD = C$

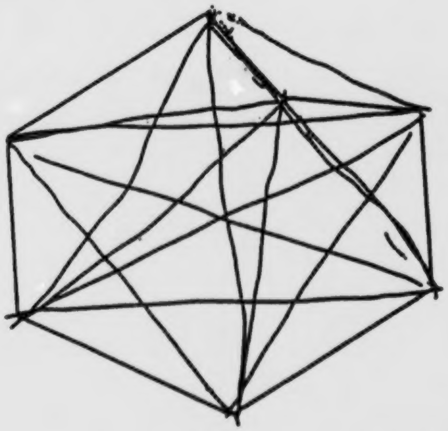
We find that $B = A(A-3)/2$ holds good.

(b) Their responses to the counterexample and their improvement of the solution.

The counterexample was proposed by SM & SH



b-1:	SM's response and improvement
b-1-1:	SM's response to the counterexample
SM	"In a heptagon, there are 14 diagonals?"
Observer	"Can you find 14 diagonals?"
SM	(After counting diagonals) "Yes, 14 diagonals."
Observer	"Does the line overlap the side?"
SM	"Yes, but..."
Observer	(Indicating the line overlapping the side) "Is this a diagonal?"
SM	"Of course, it is a diagonal."
Observer	(Indicating the line overlapping the side) "What is this?"
SM	"It is the diagonal, so it may be a diagonal at least."
b-1-2:	The counterexample written by SM.



b-1-3: SM's improvement of their solution

The improvement of the way how to find the number of diagonals.

No improvement!

b-2: SH's response and improvement

b-2-1: SH's response to the counterexample.

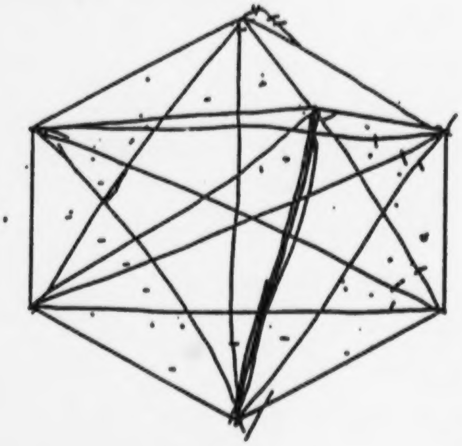
Observer "The number of diagonals ought to be 14 by your formula."

SH (SH draws some diagonals, but he doesn't draw the diagonal out of the figure or not one overlapping the side. Then, he counts the number of diagonals and triangles.) "12 diagonals"

Observer "The result doesn't match your formula, does it?"

SH "Yes, it does."

b-2-2: The counterexample written by SH



b-2-3: SH's improvement of their solution

The improvement of the way how to find the number of diagonals.

The method: If the number of angle is a, we can find it by (a 2 –3a)+ 2

Remark: The degree of each internal angle is less than 180.

What I regard as an internal angle is more than 180.

The improvement of the reason why the way is correct.

No Improvement.

(c) The response to four questions by SM & SH

c-1: The response by SM

Question 1: If you meet the case where something doesn't hold good, what do you think about that case?

SM: (By word of mouth) It is a mistake of the problem.
(By writing) "It is a mistake when making the problem."

Question 2: What is the explanation as the solution of mathematical problem?

SM: (By word of mouth) "Teaching why I can have confidence."
(By writing) Getting others to know why I can have confidence.

Question 3: If the statement has been proved, is the verification of more cases gives the problem solving more certainty?

SM: (By word of mouth) "It gives more certainty."
(By writing) It gives more certainty.

Question 4: If there is a case which doesn't satisfy the statement proved, what does it mean?
SM: (By word of mouth) "The proof was incorrect."
(By writing) It is a mistake of the proof.
c-2: The response by SH
Question 1: If you meet the case where something doesn't hold good, what do you think about that case ?

SH: (By word of mouth) "The explanation don't hold good."
(By writing) If the case where something doesn't hold good occurs, then the part which doesn't hold good. That case means that the person who said so is incorrect.

Question 2: What is the explanation as the solution of mathematical problem?

SH: (By word of mouth) Grasping the reason in detail such case that this implies it."
(By writing) Making the reason why the answer is correct more detail and more certain. Because that something doesn't hold good, although it holds good in some cases, means the way of answer isn't enough."

Question 3: If the statement has been proved, is the verification of more cases gives the problem solving more certainty?

SH: (By word of mouth) " It gives more certainty."
(By writing) It gives more correctness, because it is possible that it holds good

in some cases, but it doesn't in other cases.

Question 4: If there is a case which doesn't satisfy the statement proved, what does it mean?

SH: (By word of mouth) "It holds good in something limited."
(By writing) It is incorrect if it doesn't hold good in some cases. But, after proving it, when the other case occurs, it means we actually couldn't prove it.

Discussion of problem solving with computer in Working Group I

Problem solving is a central focus of the mathematics curriculum. An effective approach in solving problems is provided by the handheld calculator and computer, the self-conscious ability to know when and why to use one's own appropriate procedures. This research indicates that many executive procedures can be learned, resulting in significant improvements in problem solving performance. Effects can be obtained with interventions as simple as holding the procedures that focus on problem solving, and by explicitly and frequently posing questions such as "what are you doing?", Why are you doing it?" and " How will it help you?"

In mathematics, understanding cannot generally be achieved without active participation in the actual process of mathematics: in conjecture and argument, in exploration and reasoning, in formulating and solving, in calculation and verification. Calculators and computers are like "fast pencil", so the mathematical process can be made more useful and efficient than with paper and pencil. Instruction based on calculators and computers has, therefore, the potential to enhance more understanding than does traditional instruction. Calculators and computers also appeal to teachers because they introduce excitement and inventiveness to otherwise routine courses. Technology must be used when it can enhance the teaching and learning of problem solving.

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Appendix Table 1 Scoring Criterion

Shape of quadrilateral	A shape of triangle ABC	score
parallelogram	all kinds	5
	special triangle	3
rectangle	isosceles & equilateral only	5
	isosceles & equilateral & others	4
	isosceles & others	4
	isosceles only	3
	equilateral & others	4
	equilateral only	3
rhombus	right triangle (angle B= 90)	5
	right triangle	3
square	right isosceles angle B=90, AB=BC	5
	right isosceles	3
	error	1
	no answer	0

Discussion of Professor Nohda's paper:

Demana: Thank you very much. We will now open the discussion. Does anyone have any comments or questions? Professor Miwa.

Miwa: My question regards page five, fourth line from the bottom. It refers to the control group of 40 students in seventh grade. Isn't this the eighth grade, because otherwise there is discrepancy between tables 2 and 3.

Nohda: Professor Miwa is correct, it is the eighth grade.

Demana: Professor Teague.

Teague: When the students used Cabri-geometry to answer this first question, did they have any way of justifying their answer, or do you think they just experimented until they saw what they thought the pattern was and then reported that result? It seems to me that if you do the problem without the computer, you have to have some reasoning processes that would almost be a justification for the answer, but by using the computer you could arrive at the answer and not really have any sort of rationale or justification - it's just that you tried it and it worked that way.

Nohda: In my presentation I didn't explain how it is evaluated. The evaluation goes as follows: the point values go from zero to 5, and zero means there's no answer, 1 means the answer is wrong, 3 means they gave a conjecture but no proof, and 5 means that both the conjecture and proof were correct. And so if it's insufficient, then the points are in between.

Demana: Additional comments or questions? Jon.

Choate: Did you in any way have the students explain how they used the technology to solve the problem? That is, before they actually sat down and started playing, did you ask them to give an experimental design of how they were going to approach the problem?

Nohda: This is something new to Japan also, so there isn't much data; but, in this case, the students spent one whole week after school learning how to manipulate or operate this software and then for one hour in regular class they worked on this software given similar problems.

Choate: I am beginning to get the feeling that the definition of an open-ended problem is quite different for the United States and for the Japanese. The notion of having a problem that has a lot of approaches but a correct answer is not necessarily what I consider to be an open-ended problem, and let me just be specific in terms of what we just saw. If you take the figure as built and ask what can you tell me about the figure, one of the things the students would notice is that you've got a different rectangle; depending on the triangle, you've got a different inner shape. But there are lots of other questions that they could also ask about area and things like that. I wonder if we're teaching different things when we teach problem solving? I guess I think we do.

Nohda: Certainly in this particular case you cannot call this open-ended. This is just a beginning and from this one you can develop it to an open-ended type problem.

Becker: I'd like to make a comment in response to Jon's question. The question concerns what is the Japanese concept of the open approach in teaching mathematics and maybe some of our Japanese colleagues would like to comment on this and, especially, correct me if I'm wrong. Basically I think there are three components to the open approach. One component is when you have a problem that has a unique solution, but there may be many different ways (solution paths) to get that solution. Another component is when you have a problem that indeed has several or many different correct answers (or solutions). Another component is when you have a problem that might be solved in many different ways but the problem has the characteristic that other problems can be posed or formulated following the solution of the first problem. The Japanese may have used the terminology "developmental approach" in this case. There are at least these three components that I believe make up the Japanese idea of the open approach. These are addressed in Professor Shimada's book on the open approach and probably in some other publications also; e.g., Takeuchi and Sawada's book.

Nohda: That is correct.

J. Wilson: As a comment, I see another possibility in this problem other than whether it's open-ended or not by our definition. It seems to me that the potential here is what if I take this thing and change the conditions so there's a bigger mathematical problem in this other than the specific problem. I really think it's quite pretty. It's very nice.

Becker: I wonder if Professor Nohda might take a little time and describe Cabri-geometry a bit more? Where does this software come from and other background?

Nohda: Please look at page 2, for information about Cabri-geometry. Please read it for more information, okay? We enjoyed the Cabri-geometry software which Dr. Jean-Marie Loborde mentions and which means geometrical figures that can be used for teaching and learning geometry. Cabri geometry dealing with point-line relations and allows the user to realize geometrical constructions due to internal structure of the software, for the possibility to move around any of the basic point of a figure by direct interaction with the mouse. The user continuously sees a figure, indeed all it's initial properties; this best figure of coverage makes for the user the points of the figure not the static drawing, but the central object by geometrical correlations. The user can manipulate figures freely. Cabri-Geometry is a micro-world. Even if there is no standard definition of what a micro-world is. Also I would probably agree on the following relations. A micro-world is often a computer based environment.

Demana: If it's not out of order, can the chair ask a question? Your last paragraph on page 14, I think, is an eloquent defense for the use of technology in mathematics. I would simply take the last sentence where it says "Technology must be used when it can enhance the teaching and learning of" and insert "almost anything" and use that as a support for the use of the calculator from the low end all the way up. I would like to hear either the speaker or anyone else respond to that challenge.

Nohda: I would say that technology must be used when it can enhance the teaching and learning of problem solving.

Becker: I would agree essentially with the spirit of that comment, but one of the things that goes through my mind, as we talk about the use of software and technology in teaching and learning mathematics, is how can the software enhance effectively teaching and learning mathematics in the curriculum? One of the key questions may be selection of the proper software to use at the right time in a mathematics lesson. Then we have to ask questions like "To what extent can potential users like teachers make those decisions?" and "How far are we from that objective?"

Teague: I think I disagree with Jerry. The question isn't quite like where do we use it at just the right time, but thinking of software it comes to whether we _______ and how do I approach and think about mathematics if I have this as a tool any time I choose it...not just during a one hour class when I'm doing one particular problem that I can use the software on. How does my thinking about mathematics and the way I approach problems change if the software or the calculator is my constant companion?

J. Wilson: I was going to add that I think the spirit of what is exciting to me about having a variety of tools available, not only the way that Dan phrased it which is very nice, but it raises questions of what and how can we think about mathematics differently given these tools. What mathematics can we now approach in the schools? How do we develop thinking that's basically different now that these tools are available? It opens up lots of new opportunities and I think all of us have to think about and work on those things and it's not an easy task. I think that you could look at the careers of Demana, Waits, Fey and Dugdale and those which think constantly about these kinds of tools and making them available to the rest of us in education.

Choate: I heard a quote by Richard Feynmann, a physicist, that really helped me understand problem solving. This quote was when he was asked "How do you learn problem solving?" He said the first thing you have to do is listen to what the problem is trying to tell you, before you tell the problem all the mathematics you know. And when you think about that I think that really says something about what technology we have now. Technology now allows us to listen to problems and watch as people play with Cabri-geometry or Sketchpad or I think what we're seeing is the beginning of the truth of that statement that you now have the ability to play with problems, to listen to problems in a way we never have before. And I think we're just beginning to understand what that means and what we do with that as educators. But I don't think we can forget that the very sort of playful aspect that we're talking about educating children and children like to play. I think this is very important, the ability that the technology allows us to play. I hope we don't forget that.

Damarin: I don't want to disagree at all with the spirit of what's been said but perhaps I take a more cautious approach to some of these things. Joseph Weisenbaum was the first person that I know of that made the observation that when you give a child a hammer, suddenly everything looks like it needs hammering. And I think that we might be erring in that direction. I think we have to always be concerned with that question. I also think that Mr. Sawada's questions this morning were very important. As we look at what these technologies are doing to our curriculum as we enter into that playfulness, and get swept up in it, maybe we forget what we're doing.

Becker: I would like to hear more from the strong proponents of this use of software, as to how the role of the teacher changes in the mathematics classroom, so that these good aspects of the use of the software will benefit all of the mathematics students in the classroom.

Teague: The first thing that happens is the teacher is no longer the fountain of all questions. They aren't the holder of the questions any more because students are now able to ask their own questions. And that seems to me to be half of mathematics. For a long time mathematics has been the answering of questions, but a large part of mathematics and mathematical understanding is knowing what questions to answer, what questions to consider. That's the most obvious first change in the teaching - that students now ask the questions.

Dugdale: Another aspect of it is that we traditionally have students responding in class with an expectation that they will be told that they are either right or wrong. With the implementation of computers, students often enter a response into the computer to see what effect their response has on a mathematical model, rather than to be told authoritatively whether they are right or wrong. I think this change of interaction modes has lots of implications.

J. Wilson: Backing away from just the technology, I think we are to be held accountable, we teachers, for anything that's going on in the classroom. Heaven knows that most of the stuff that I see out there that's in classrooms is deadly. Kids can't draw graphs because it is so tedious, boring and overwhelming that they never learn. Somebody should be held accountable for that disaster that's going on. If we introduce a new piece of software, certainly we should be held accountable for what happens with it, but let's don't say, don't use software; let's keep doing the stuff we've been doing, we don't have to defend the old stuff, but anything that's new we do have to. I think we have to be accountable for everything that comes along.

Sawada: I think I may speak for many teachers not necessarily the people who are here today, however. The focus on problem solving as opposed to, say, word problems per se seems to be something that has arisen since the introduction of technology and computers to the classroom. Is it reasonable to think of the word problem and problem solving as being separate, or are they aspects of the same sort of thing?

Fey: I don't know that I'm the best spokesman to respond to that, but I certainly would disagree with the notion that problem solving has been a result of technology or even an emphasis on it. If anything, in this country, the impact of technology has been given a boost by a sort of a new awareness of the role of problem solving; but I'm old enough to remember that we wrote about problem solving in the 1950s as well. And there were people working in that area, historically, way back. I don't find problem solving as distinct from word problems or something that uses technology, and I certainly work on problems which I don't consider word problems and don't use computers. I would add also that some of the things that I like to do with the computer, in playing with it and in playing with problems, involves exploration and when I'm ready to turn the computer off, I haven't solved the problem, I've created a problem. I've got something new. Then there's the demonstration to be done or an exploration that's away from the computer. So, the role isn't just in problem solving, it's one of problem generating or exploration and finding new areas.

Demana: Perhaps in the interest of staying on schedule please join me in thanking our speaker, Professor Nohda, and get ready for the next one so we don't get too far behind. Thank you very much.

End of Discussion

MATHEMATICS INSTRUCTIONAL SOFTWARE IN JAPANESE CLASSROOMS

Shin'ya Sakitani
Hyogo University of Teacher Education
Hyogo, Japan

Shin'ji Iida
Fukuoka University of Education
Fukuoka, Japan

1. Instructional Software in Mathematics Classrooms

We can divide the mathematics instructional software often used in Japanese classrooms into the following three broad categories : drill and practice, demonstration of mathematical idea and process, and presentation of learning situation.

Drill and Practice

Computer-based drill and practice makes it possible for students to execute them at their own paces. Students can get an immediate feedback or diagnostic comment from the computer when they make an error. The computer is more effective and efficient than a teacher in diagnosing and correcting the student's error.

Records of the students' responses are stored in the computer. The teacher can make use of them to evaluate their understanding and progress. The teacher also gets some suggestions for further teaching from this information.

Demonstration of mathematical idea and process

We can display the mathematical idea and process on the computer screen. For example, we can display some cutting planes of a cube on the screen. Another example is to show the process which converges into a rectangle as we divide a circle into smaller fan shapes and put them side by side. This is the process to find the formula for the area of a circle. Though students can carry out these operations, actually it might be too time-consuming and sometimes dangerous. The computer carries them out efficiently instead of students. The purpose of these software is to facilitate the students' understanding of mathematical idea and process.

Presentation of learning situation

It is necessary to prepare the appropriate learning situation for successful learning. If students get such a situation, they will begin to explore the situation and analyze it mathematically. The computer can present such a situation by giving the data necessary to learning and sometimes by taking the form of a game. Students might start their mathematics learning for themselves from these data or games.

Of course, like the case of demonstration of mathematical idea and process, students can actually collect the data from their daily lives or do the game by themselves. But, in these cases, it might also be time-consuming and it is possible for a computer to present not only a real life situation but also a situation which might exist.

Some examples of these instructional software will be illustrated in the following section. But, these software only make students receive the information from the computer. When we explore the effective use of the computer, we must develop the software which requires more active involvement of students with the computer. In such software, students can act on a computer and make use of it as a tool for doing or exploring mathematics. So, we call this kind of software "exploration software." Examples of such software are THE GEOMETRIC SUPPOSER and THE GEOMETER'S SKETCHPAD which were developed in the U.S.A. So as not to bring a computer to an end as "electric picture-card show," we must strive to develop software of this sort.

Though it is difficult to find exploration software in Japan, we were able to find software of this kind which was developed by a Japanese senior high school teacher. We will present this software in the latter half of this paper. But, this kind of software is not so popular in Japanese classrooms. Almost all software which are used in Japanese classrooms belong to the above-mentioned three categories.

2. Some Examples of Computer Use in Japanese Classrooms

(1) Computer Use as Drill and Practice

Needless to say, the purpose of computer use as drill and practice is mastering various skills. By using the computer as a "data bank" of problems, many software for learning enable us to treat the following three matters which are indispensable for mastering skills:

(i) To be able to select the skill which should be mastered. (We can usually select that on the screen of the main menu.)

(ii) To be able to select the level of the skill which should be mastered. (We can usually select that on the screen of the sub-menu.)

(iii) To be able to tackle a number of exercises which are successively displayed by the computer, for the purpose of mastering the selected skill. Some exercises are displayed by the computer, and others are inputted by the learners.

Software for Arithmetic Learning by Osaka Publication Company

Dropping Balls	Grade 1	Area of Circle	Grade 5
Flash Cards	Grade 1-3	Extended Figure	Grade 6
Clock	Grade 1-2	Drawing Pictures	The Lower Grades
Traffic Survey	Grade 3	Game of Labyrinths	The Higher Grades
Circle	Grade 3	Dice	All Grades
Game of Pitching Camps	Grade 4	End	

figure 1

a. Dropping Balls (Grade 1)

Purpose : Mastering skills of joining and separating numbers from 1 to 5

Method of use : Select "Dropping Balls" on the screen of the main menu, then the sub-menu such as is displayed in figure 2. On the screen of this sub-menu, select "joining" or "separating."

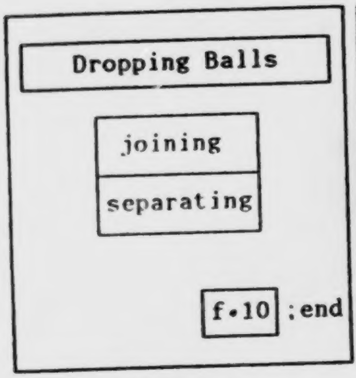

Dropping Balls
joining
separating
f•10 ;end

figure 2

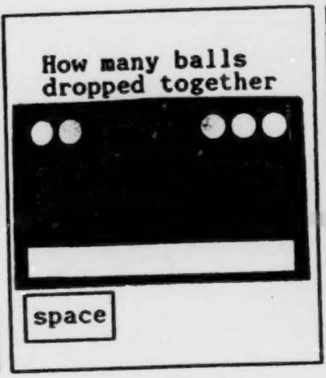

How many balls dropped together
space

figure 3

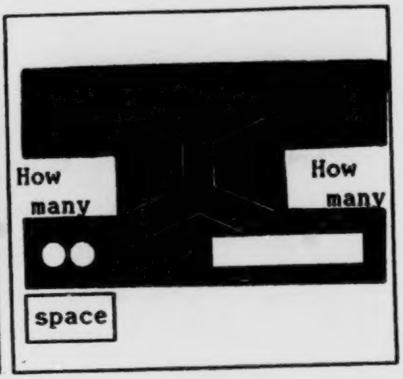

How many
How many
space

figure 4

An example of "joining"

1. "How many balls do you put in the left side?" "2"

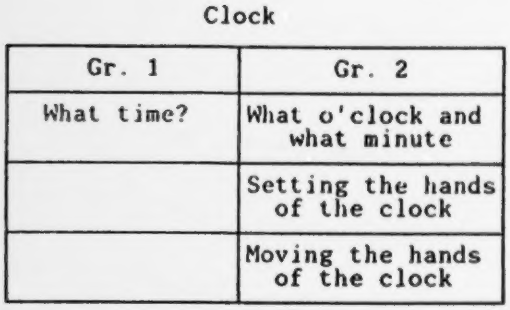
2. "How many balls do you put in the right side?" "3"
3. Figure 3 is displayed on the screen.
4. "space"; These balls are dropped in the lower box.
5. "space"; The balls in the lower box come to be seen.
6. "space"; The sub-menu can be displayed again.

Figure 4 shows an example of "separating".

b. Clock (Grade 1-2)

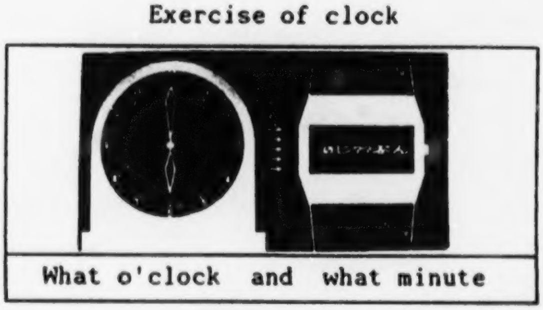
Purpose: Mastering skills of telling the indicated time and setting the hands of the clock for the indicated time.

Method of use: Select "Clock" on the screen of the main menu, then the sub-menu such as figure 5 is displayed. Though the learning content of "Clock" between Gr.1 and Gr.2 are distinguished on this screen of the sub-menu, our Course of Study says that first graders should learn most of this content.


Clock

Gr. 1	Gr. 2
What time?	What o'clock and what minute
	Setting the hands of the clock
	Moving the hands of the clock

figure 5


Exercise of clock

What o'clock and what minute

figure 6

An example of "What o'clock and what minute"

Input the time which the analogue clock shows by punching ten keys on the keyboard. (see figure 6) For example, when the analogue clock shows 8:30, we input "8" "30". If the answer is correct, a mark of a correct answer is displayed under the digital clock. Even though the answer is wrong, the learners can try to input up to three times. If the learners make errors three times, a correct answer is displayed.

In the activities of "Setting the hands of the clock", pupils can set the hands of the analogue clock for the indicated time by the digital clock.

(2) Computer Use as Demonstration of Mathematical Idea and Process

The purpose of computer use as demonstration of mathematical idea and process is mainly helping the learners understand certain knowledge. Visualization plays an important part in understanding knowledge; moreover, by using computers, we can visualize certain knowledge dynamically. Though we could not help demonstrating certain knowledge by using concrete objects such as teaching aids until now, we think the simulation by computers promotes understanding of knowledge.

a. Area of Circle (Grade 5)

Purpose: Understanding the area of circles

Method of use: Select "Area of Circle" on the screen of the main menu, then the sub-menu such as figure 7 is displayed. "Area of Circle 1" demonstrates a circle which is divided into many sectors and transformed into a parallelogram. "Area of Circle 2" demonstrates a circle which is divided into many concentric rings and transformed into an isosceles triangle.

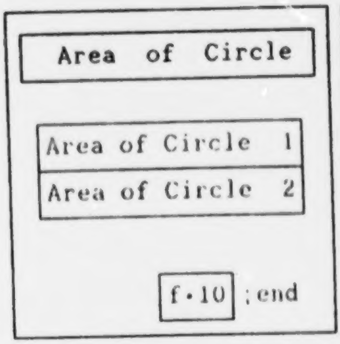
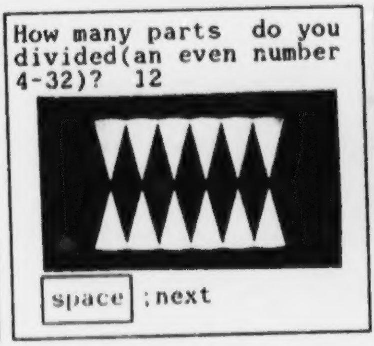
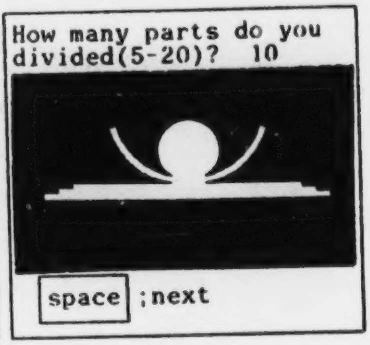


figure 7



How many parts do you divided(an even number 4-32)? 12

figure 8



How many parts do you divided(5-20)? 10

figure 9

An example of "Area of Circle 1"

1. Input the number of dividing sectors (an even number 4-32). Then, the circle is divided into two equal parts and further divided into sectors according to the number inputted.
2. "space"; Two half-circles are partitioned into the parts like saws, and stopped such as figure 8.
3. "space"; Two parts like saws are joined.
4. "space"; "radius" and "half of circumference" are displayed as measure.

Figure 9 shows an example of a screen included in "Area of Circle 2"

(3) Computer Use as Presentation of Learning Situation

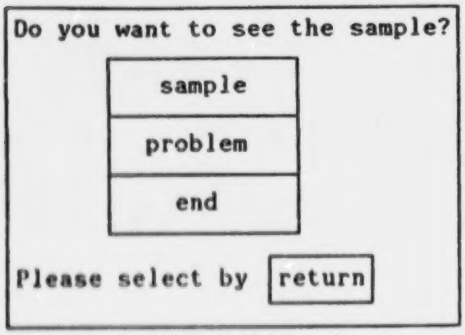
Though the computer use as drill, practice and demonstration is typical in Japanese arithmetic-mathematical education, the activities tackled by computer use cannot necessarily be regarded as problem solving. So, let's introduce software for presentation of learning situations which is suitable as problem solving through the use of computers.

Since these are also displayed through certain simulations, it is difficult to distinguish this type from the type (2). But, we think, this type of computer use gives impetus to generate mathematical ideas from presented learning situation or apply these to problem solving. Then, we can distinguish this type of computer use from the type of (2) which demonstrates directly mathematical ideas and processes themselves.

a. Traffic Survey (Grade 3)

Purpose: Being able to apply lists and graphs to data reduction for the traffic survey

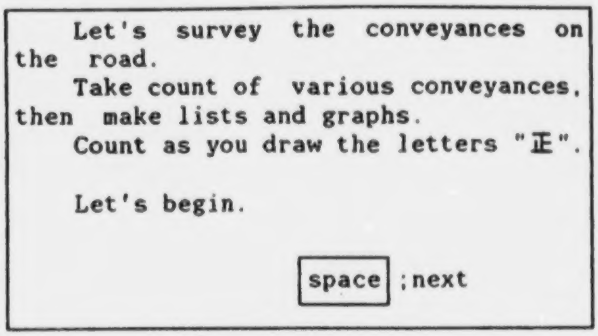
Method of use: Select "Traffic Survey" on the screen of the main menu, then sub-menu such as displayed in figure 10.



Do you want to see the sample?

Please select by return

figure 10



Let's survey the conveyances on the road.

Take count of various conveyances, then make lists and graphs.

Count as you draw the letters "正".

Let's begin.

figure 11

Next, a screen for explanation (figure 11) is displayed. But, in using this software as an introductory learning situation, we had better ask the learners such questions as follows:

"What kind of and how many conveyances on the road?"

"How do you record and represent?"

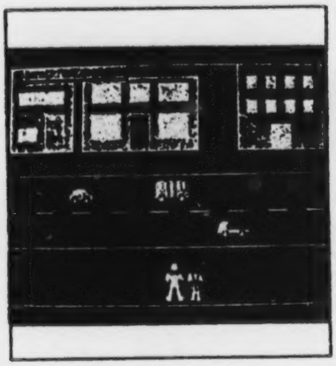
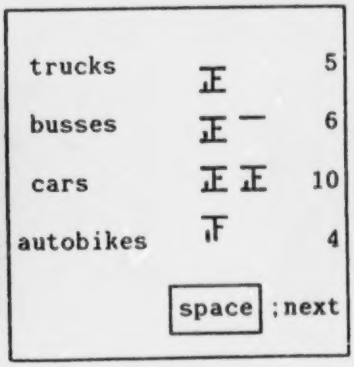


figure 12



trucks	正	5
busses	正 一	6
cars	正 正	10
autobikes	下	4

figure 13

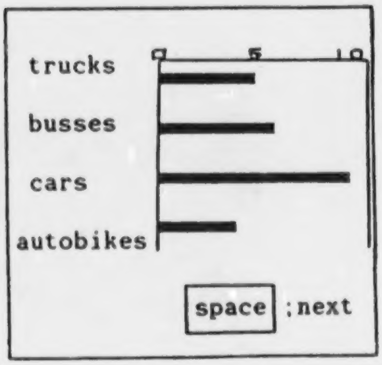


figure 14

1. "space"; Four kinds of conveyances (trucks, busses, cars and autobikes) start passing on the road on the screen as in figure 12. The learners record the number of these conveyances without exception and duplication, as they watch this screen.
2. "space"; The right list of recording such as figure 13 is displayed.
3. "space"; The bar graph such as figure 14 is displayed.

b. Game of Pitching Camps (Grade 4)

Purpose: Introducing the method of requiring area of rectangles through the game of pitching camps

Method of use: Select "Game of Pitching Camps" on the screen of the main menu.

1. "space"; The first camp of the player A is determined by computers at random (see figure 15).
2. The player B selects one of the camps by cursor keys and determines it by the return key. Both players must determine the new camps adjoining any camps which have been already pitched by themselves.
3. When either player A or B cannot pitch any camps, the game is over. By means of asking the learners such a question as follows : "How do you determine which player won?", we would like to make the learners turn their attention to the area. According to the result of the game, it is possible that we cannot easily determine which player won. In this case, we would like to make them realize the necessity of introducing the measure like a section paper.
4. "space"; The measure is introduced (see figure 16). The learners take count of measure by themselves.
5. "space"; The computer displays the sum of the measures which there are in the camps pitched by both players.

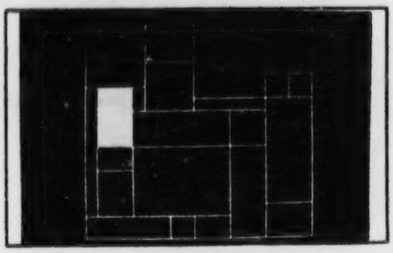


figure 15

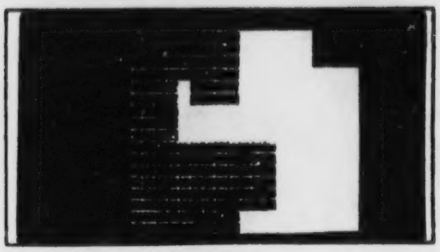


figure 16

3. ANALYTIC SUPPOSER

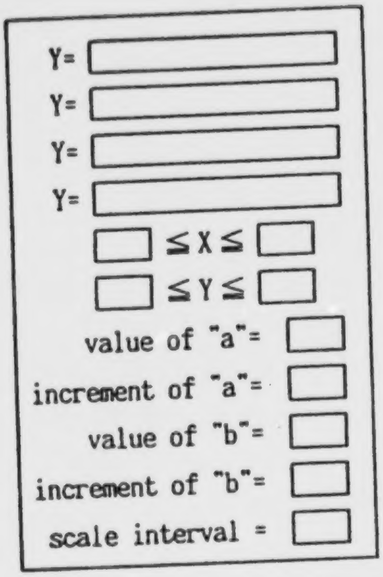
(1) The contents of the software

We give an outline of exploration software which we found in Japan and we will illustrate how this software is used later.

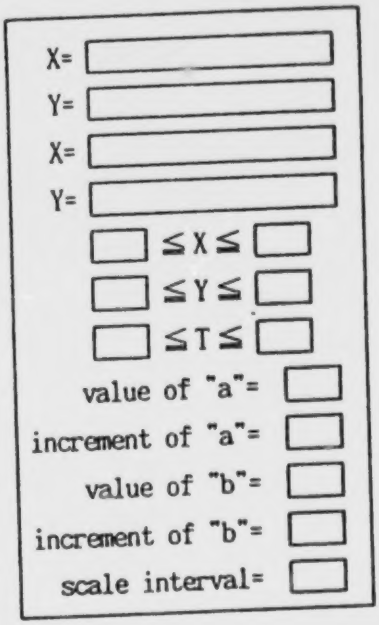
This software has three kinds of screens. These are the following three types:

On the screen of type A, we can draw the graphs of functions by typing the expressions of functions whose independent variable is X. We can change the domain of X and Y freely. If the expression of function contains the letter "a" or "b", we can change the value of "a","b" regularly according to the increment of "a","b", and we can draw the axes of coordinates in any scale interval. As the color of expression corresponds to the color of graph, it is easy to discriminate the graphs.

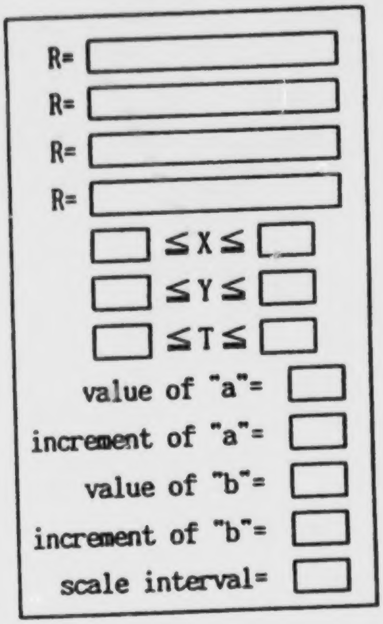
Type A


Y=
Y=
Y=
Y=
≤ X ≤
≤ Y ≤
value of "a"=
increment of "a"=
value of "b"=
increment of "b"=
scale interval =

Type B


X=
Y=
X=
Y=
≤ X ≤
≤ Y ≤
≤ T ≤
value of "a"=
increment of "a"=
value of "b"=
increment of "b"=
scale interval=

Type C


R=
R=
R=
R=
≤ X ≤
≤ Y ≤
≤ T ≤
value of "a"=
increment of "a"=
value of "b"=
increment of "b"=
scale interval=

We can extend the domain on the screen twice or reduce it half. So, if we continue this operation, we can extend or reduce the domain on the screen as large or small as we want (zoom -out or zoom-in). Furthermore, we can move any point on the screen to the center of screen.

These operations are executed easily by pushing the following function keys.

"f·1"	draw the graph
"f·2" or "Rollup".........	increase the value of "a" with every increment of "a"
"f·3" or "Rolldown".....	decrease the value of "a" with every increment of "a"
"f·4"......................	increase the value of "b" with every increment of "b"
"f·5"	decrease the value of "b" with every increment of "b"
"f·6"	extend the domain on the screen twice
"f·7"	reduce the domain on the screen half
"f·8"	display the sign + (and we move it by "→", "←", "↑" and "↓" keys, then draw the graph whose center is that point by "return" key)
"f·10"....................	change the type of screen

On the screen of type B, we can draw the parametrized curves whose parameter is "T". On the screen of type C, we can draw the curves of polar equations which involve the variables "R" and "T". The functions of function keys are the same as the case of type A.

(2) The purpose of the software

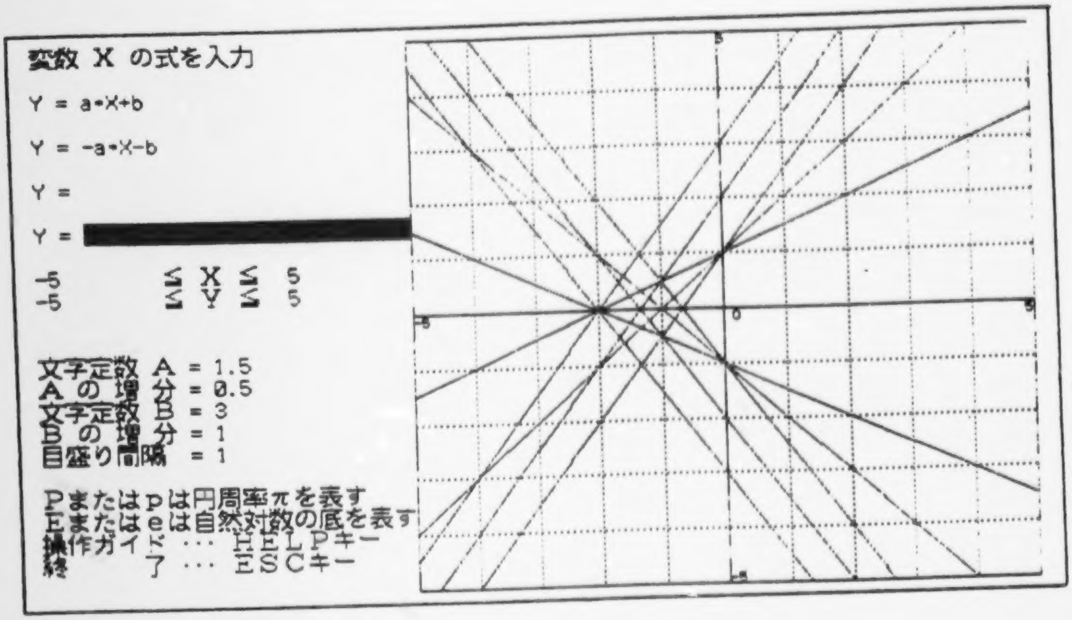
Students sitting in front of the computer input some expressions of functions and draw their graphs first. They will conjecture the properties of functions observed from graphs. Their conjectures can be quickly tested by drawing the graphs of other functions. When their conjectures hold with other functions, they want to prove them somehow. This gives a good motivation for a proof. If they find counterexamples for their conjectures, they abandon them immediately and look for other properties. We consider these activities as the process to do or explore mathematics.

The purpose of this software is to make students derive their conjectures about properties of functions and elaborate them. At this process, this software allows students to find examples or counterexamples for their conjectures easily. This is the case of using the computer as a tool for doing or exploring mathematics. So, we name this software "ANALYTIC SUPPOSER" by torturing THE GEOMETRIC SUPPOSER. We hope that this software will give new ways for students to learn analysis at the secondary school level.

(3) Some Examples of How to Use ANALYTIC SUPPOSER

We will show some examples of how ANALYTIC SUPPOSER is used. The contents of the following illustrations, however, should be the results of students' trials and errors. Students would input many expressions of functions until they got these results.

Example - 1



```
変数 X の式を入力
Y = a*X+b
Y = -a*X-b
Y =
Y =
-5         ≦ X ≦  5
-5         ≦ Y ≦  5

文字定数 A = 1.5
A の 増 分 = 0.5
文字定数 B = 3
B の 増 分 = 1
目盛り間隔  = 1

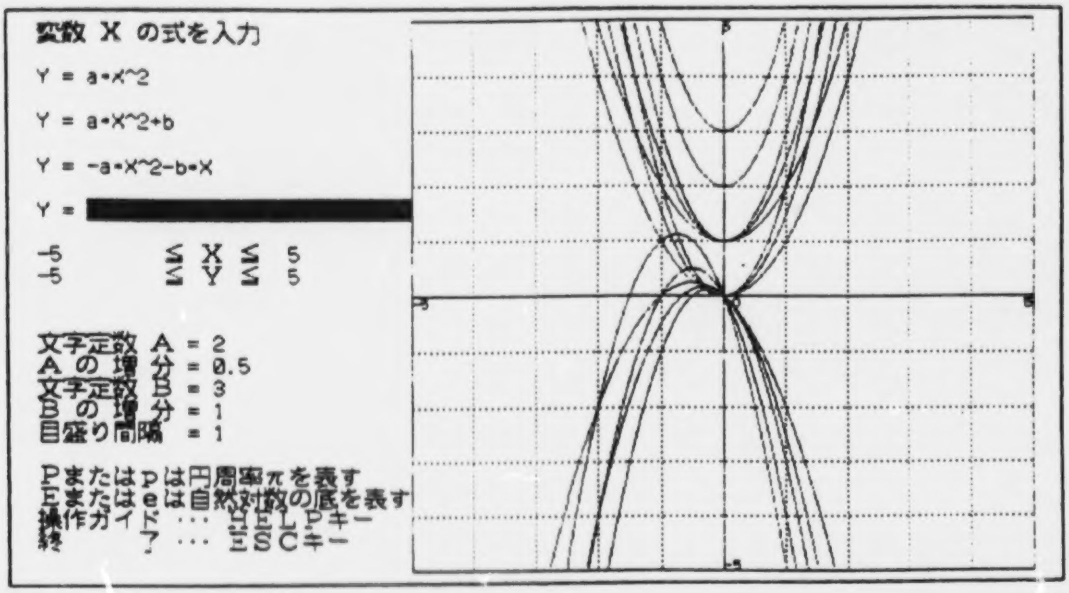
PまたはPは円周率πを表す
Eまたはeは自然対数の底を表す
操作ガイド ··· HELPキー
終     了 ··· ESCキー
```

Students can find some properties of linear function by changing the value of the coefficient of x and constant term. For example, they will find the following properties by drawing the graphs of y=ax+b, y=–ax–b for some values of "a" and "b".

1) If the coefficient of x is positive, the graph has a positive slope and slants upward from left to right; if the coefficient of x is negative, the graph has a negative slope and slants downward from left to right.
2) A line with large positive slope rises faster than a line with small positive slope; a line with small negative slope is steeper than a line with large negative slope.
3) The graphs of these functions are lines with the same slope or parallel to each other when the value of the coefficient of x is equal.
4) The graph of y=ax+b is a shift upward b units of the graph of y=ax; the graph of y=–ax–b is a shift downward b units of the graph of y=–ax.

From these properties, students can conjecture that the graph of the linear function y=ax+b is a straight line with slope "a" and y-intercept (0,b).

Example - 2



```
変数 X の式を入力
Y = a*X^2
Y = a*X^2+b
Y = -a*X^2-b*X
Y =
-5        ≦ X ≦  5
-5        ≦ Y ≦  5

文字定数 A = 2
A の 増 分 = 0.5
文字定数 B = 3
B の 増 分 = 1
目盛り間隔  = 1

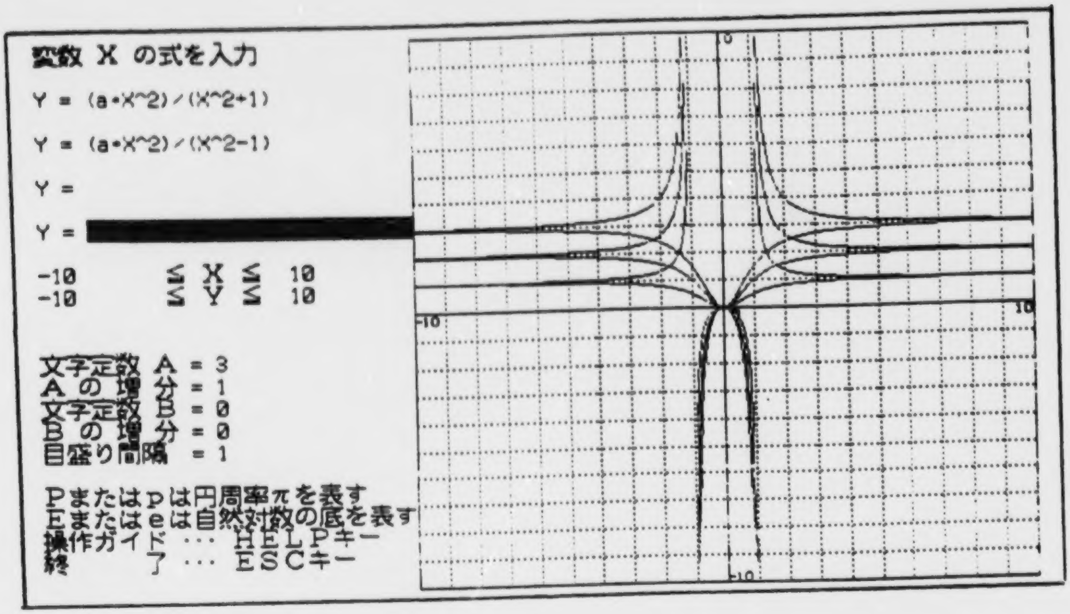
Pまたはpは円周率πを表す
Eまたはeは自然対数の底を表す
操作ガイド ・・・ HELPキー
終     了 ・・・ ESCキー
```

Students will find the following properties about a quadratic function by drawing the graphs of $y=ax^2$, $y=ax^2+b$, $y=-ax^2-bx$ for some values of "a" and "b".

1) The graph of a quadratic function is a parabola and symmetric.
2) Each graph has the same shape when the coefficients of x^2 are equal. The only difference is the position of the graph.
3) When the coefficient of x^2 is positive, the parabola opens upward and the larger the value of coefficient, the narrower the parabola becomes; when the coefficient of x^2 is negative, the parabola opens downward and the larger the coefficient, the broader the parabola becomes.
4) The value of "b" of $y=ax^2+b$ shifts the graph of $y=ax^2$ vertically, but does not change its shape and its axis of symmetry.
5) The value of "b" of $y=-ax^2-bx$ changes the axis of symmetry of $y=-ax^2$, but does not change its shape.

From these properties, students may conjecture that different values of "a","b", and "c" of the quadratic function $y=ax^2+bx+c$ affect the shape of the parabola, the axis of symmetry and the y-intercept, respectively.

Example - 3



```
変数 X の式を入力
Y = (a•X^2)/(X^2+1)
Y = (a•X^2)/(X^2-1)
Y =
Y =
-10      ≦ X ≦  10
-10      ≦ Y ≦  10

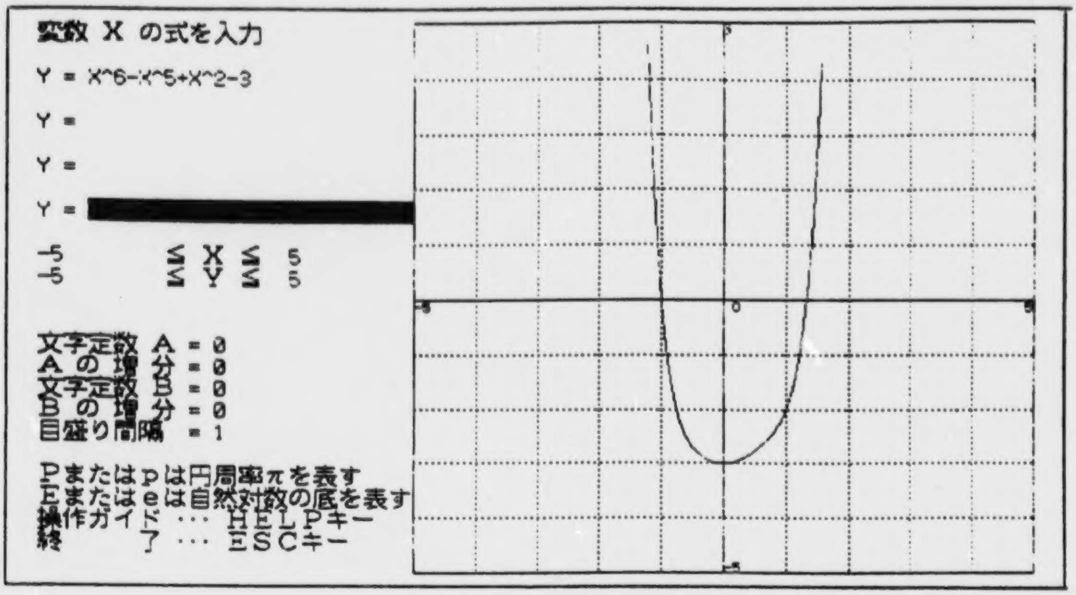
文字定数 A = 3
A の 増 分 = 1
文字定数 B = 0
B の 増 分 = 0
目盛り間隔 = 1

PまたはPは円周率πを表す
Eまたはeは自然対数の底を表す
操作ガイド ・・・ HELPキー
終     了 ・・・ ESCキー
```

Students may draw the graphs of some rational functions and find some properties. For example, if the degree of the denominator is larger than that of the numerator, the value of function approaches 0 as x gets larger; if the degree of the denominator is smaller than that of the numerator, the value of the function approaches ±∞. But, if the degrees of denominator and numerator are equal, the graph of the function is rather strange. In order to explore this case, students may draw the graphs of $y=\frac{ax^2}{x^2+1}$ and $y=\frac{ax^2}{x^2-1}$ for some values of "a". They will conjecture the following properties from these graphs.

1) If the denominator is not 0 for every x, the value of the function approaches one real number as x gets larger. In fact, the line y=a is a horizontal asymptote for the graph of $y=\frac{ax^2}{x^2+1}$.
2) If the denominator is 0 for some xs, the graph of the function has some vertical asymptotes. Since x=±1 make the denominator 0 for the function $y=\frac{ax^2}{x^2-1}$, the line x= ±1 are vertical asymptotes.

Example - 4



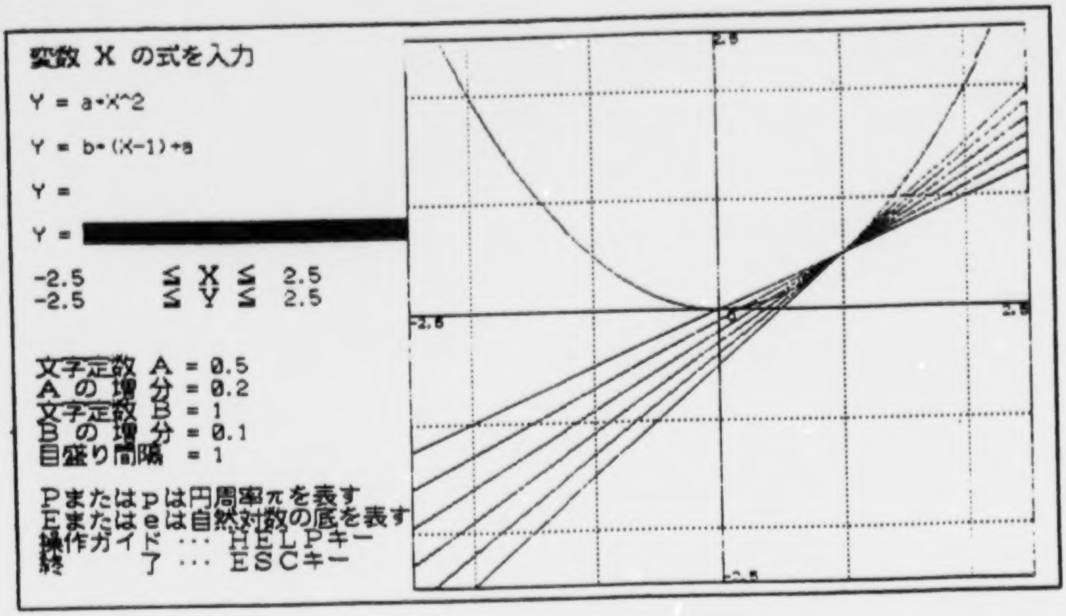
```
変数 X の式を入力
Y = X^6-X^5+X^2-3
Y =
Y =
Y =
-5        ≦ X ≦  5
-5        ≦ Y ≦  5

文字定数 A = 0
A の 増 分 = 0
文字定数 B = 0
B の 増 分 = 0
目盛り間隔  = 1

Pまたはpは円周率πを表す
Eまたはeは自然対数の底を表す
操作ガイド ･･･ HELPキー
終    了 ･･･ ESCキー
```

The zeros of a function defined by y=f(x) are solutions to the equation f(x)=0. So, if students draw the graph of y=f(x), they can get the approximate values of real zeros of this function by extending the domain on the screen and making the scale interval smaller. For example, if students draw the graph of $Y=x^6-x^5+x^2-3$, they find that there is a zero between 1 and 2. Then they can find that there is a zero between 1.3 and 1.4 by extending the domain and making the scale interval 0.1. Here, they can approximate to the nearest tenth the real zero of this function; that is, the solution to the equation $x^6-x^5+x^2-3=0$. If they want to get a more accurate value, they can repeat this procedure.

Example - 5



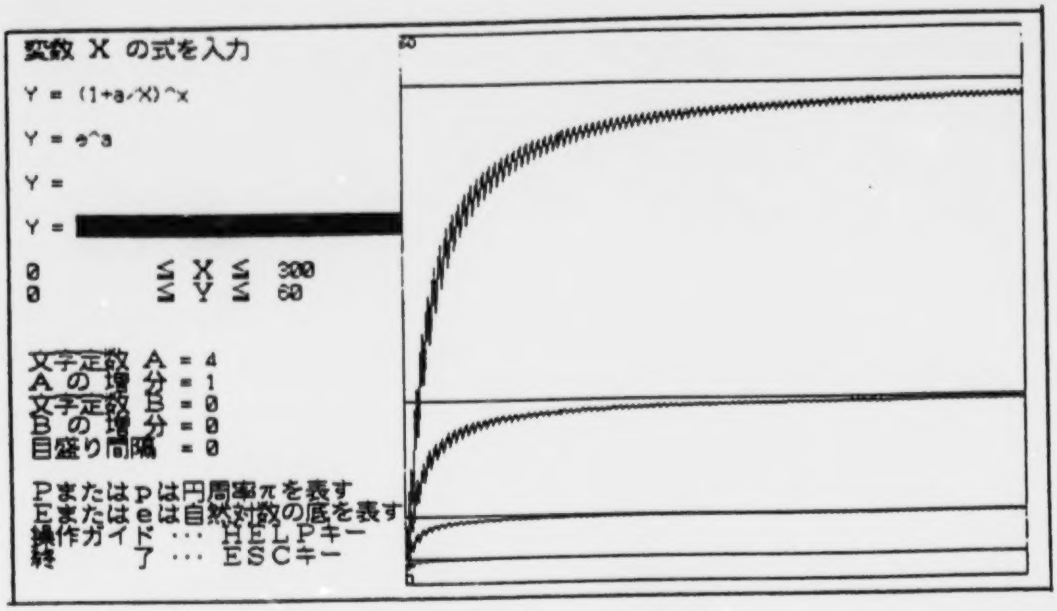
```
変数 X の式を入力
Y = a*X^2
Y = b*(X-1)+a
Y =
Y =
-2.5      ≦ X ≦  2.5
-2.5      ≦ Y ≦  2.5

文字定数 A = 0.5
A の 増 分 = 0.2
文字定数 B = 1
B の 増 分 = 0.1
目盛り間隔  = 1
Pまたはpは円周率πを表す
Eまたはeは自然対数の底を表す
操作ガイド ··· HELPキー
終      了 ··· ESCキー
```

If students draw the graph of parabola $y=ax^2$, this graph contains the point (1,a). They may draw the line $y=b(x-1)+a$ which also contains the point (1,a) and change the values of "a" and "b" in order to find the relation between the slope of this line and parabolas. From these activities, they may conjecture that the slope of the tangent line to the parabola $y=ax^2$ at the point (1,a) is 2a.

But, from the graph of parabola, the slope of the tangent line cannot be 2a always. So, they will change the point which is contained in the parabola and keep on exploring. For example, as the point (2,4a) is contained in the parabola $y=ax^2$, they change the expression of the line to $y=b(x-2)+4a$ and repeat the preceding activities. But, in this case, the slope of the tangent line is 4a. From this and previous results, they may conjecture that the slope of the tangent line to the parabola $y=ax^2$ at the point (x,ax^2) is 2ax. These activities may lead students to the concept of derivative.

Example - 6



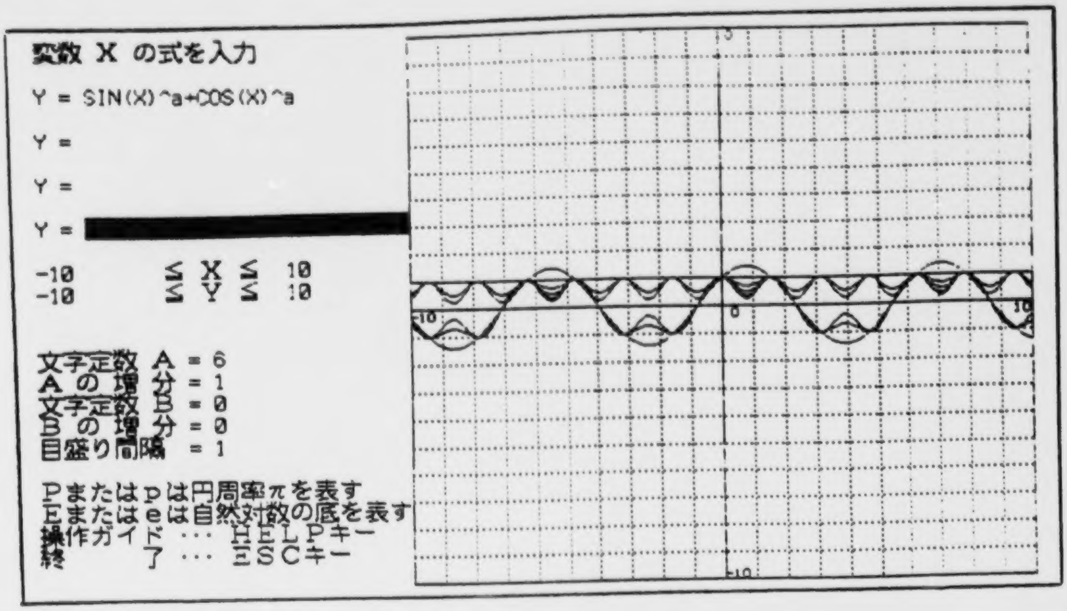
```
変数 X の式を入力
Y = (1+a/X)^x
Y = e^a
Y =
Y =
0        ≦ X ≦  300
0        ≦ Y ≦  60

文字定数 A = 4
A の 増 分 = 1
文字定数 B = 0
B の 増 分 = 0
目盛り間隔 = 0

Pまたはpは円周率πを表す
Eまたはeは自然対数の底を表す
操作ガイド ・・・ HELPキー
終     了 ・・・ ESCキー
```

If students have learned lim $(1+\frac{1}{x})^x = e$, some students want to explore the limit for a similar expression. They may draw the graphs of the function $y=(1+\frac{a}{x})^x$ for some values of "a". They will find that the value of $(1+\frac{a}{x})^x$ gets closer to some fixed number as x gets larger. They may conjecture that the limit value of this function is e^a and draw the graph of $y=e^a$ to confirm this conjecture.

Example - 7



```
変数 X の式を入力
Y = SIN(X)^a+COS(X)^a
Y =
Y =
Y =
-10      ≦ X ≦  10
-10      ≦ Y ≦  10

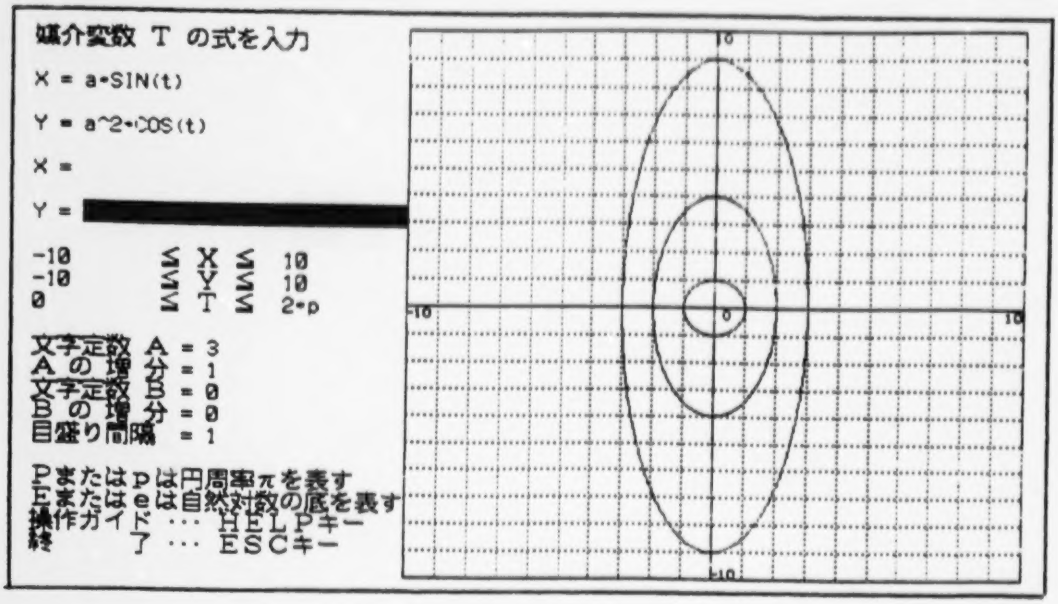
文字定数 A = 6
A の 増 分 = 1
文字定数 B = 0
B の 増 分 = 0
目盛り間隔  = 1

PまたはPは円周率πを表す
Eまたはeは自然対数の底を表す
操作ガイド ･･･ HELPキー
終     了 ･･･ ESCキー
```

Students can draw the graphs of functions which are constituted from circular functions. Some students may draw the graphs of $y=\sin^a x+\cos^a x$ for some values of "a" and find the property that $\sin^2 x+\cos^2 x=1$ for all values of x. Other students may draw the graph of $y=\frac{\sin(x)}{x}$ and conjecture that the value of this function approaches 1 as x approaches 0. Some of them will further explore the function $y=\frac{\sin(ax)}{a^2 x}$ for some values of "a" and find that the values of these functions approach $\frac{a}{a^2}$ as x approaches 0.

If students have learned $\lim_{x\to\infty} x\sin\frac{1}{x}=1$, they may try to find whether $x\cos\frac{1}{x}$ and $x\tan\frac{1}{x}$ have similar property by drawing the graphs of $y=x\cos\frac{1}{x}$ and $y=x\tan\frac{1}{x}$.

Example - 8

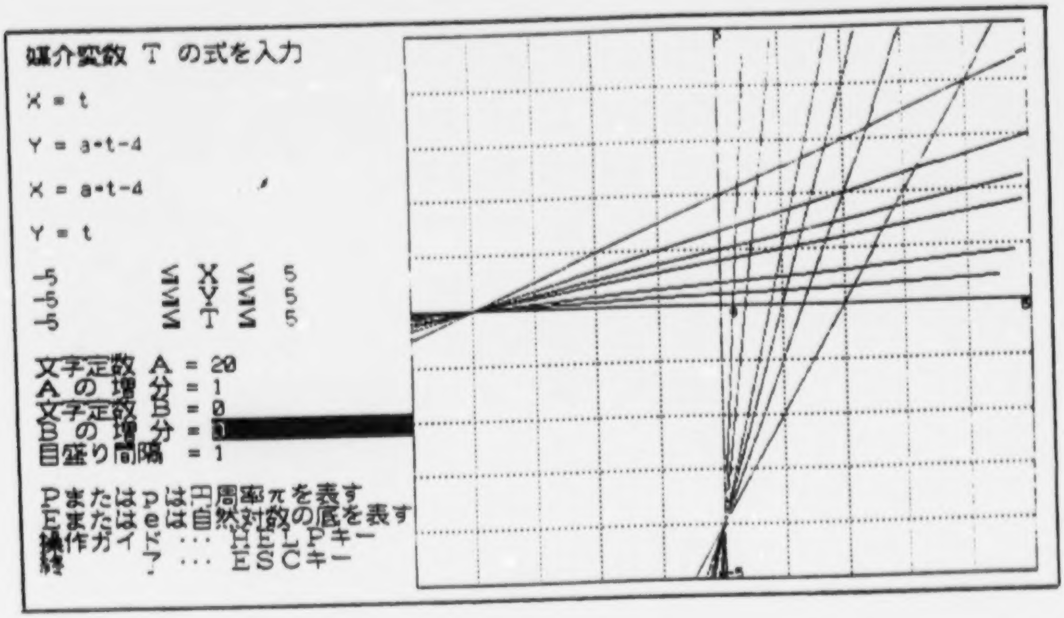


```
媒介変数 T の式を入力
X = a*SIN(t)
Y = a^2*COS(t)
X =
Y =
-10      ≦ X ≦  10
-10      ≦ Y ≦  10
0        ≦ T ≦  2*p
文字定数 A = 3
A の 増 分 = 1
文字定数 B = 0
B の 増 分 = 0
目盛り間隔 = 1
Pまたはpは円周率πを表す
Eまたはeは自然対数の底を表す
操作ガイド … HELPキー
終    了 … ESCキー
```

Students can draw the parametrized curves. If they draw the curve whose parametric equations are x=asin(t), y=a^{2}cos(t) and change the value of "a", they get a circle when a=1, ellipses when a≠1. From these data, they may conjecture that the curve is a circle when the coefficients of sin(t) and cos(t) are equal; it is an ellipse when they are different.

Some students may draw the curves whose parametric equations are x=a(t–sin(t)), y=a^2(1–cos(t)) for some values of "a". They will get a cycloid when a=1.

Example - 9

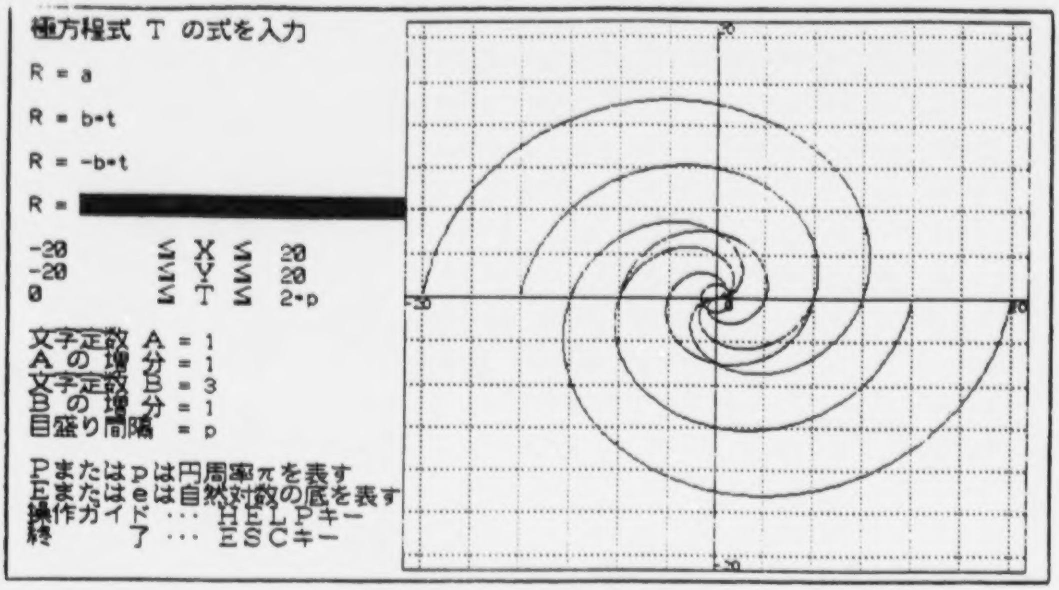


```
媒介変数 T の式を入力
X = t
Y = a*t-4
X = a*t-4
Y = t
-5   ≦ X ≦ 5
-5   ≦ Y ≦ 5
-5   ≦ T ≦ 5
文字定数 A = 20
A の 増 分 = 1
文字定数 B = 0
B の 増 分 = 1
目盛り間隔  = 1
Pまたはpは円周率πを表す
Eまたはeは自然対数の底を表す
操作ガイド ... HELPキー
終了       ... ESCキー
```

Students can get the inverse function of one-to-one function by interchanging x and y. If students draw the graphs of f(x=t,y=at–4) and g(x=at–4,y=t) for some values of "a" , they may conjecture the property that these functions are symmetric with respect to the line y=x; that is, one graph is obtained from the other by reflecting it across the line y=x. They will confirm this conjecture when they draw the graphs of f(x=t,y=at^3) and g(x=at^3, y=t) for some values of "a".

When students draw the graphs of f(x=t,y=t^2) and g(x=t^2,y=t), they will find that if the function (f) is not one-to-one, it cannot have an inverse function because g is not a function; that is, certain values of x produce more than one value for y.

Example - 10



```
極方程式 T の式を入力
R = a
R = b*t
R = -b*t
R =
-20     ≦ X ≦  20
-20     ≦ Y ≦  20
0       ≦ T ≦  2*p
文字定数 A = 1
A の 増 分 = 1
文字定数 B = 3
B の 増 分 = 1
目盛り間隔 = p
Pまたはpは円周率πを表す
Eまたはeは自然対数の底を表す
操作ガイド ･･･ HELPキー
終    了 ･･･ ESCキー
```

Students can explore the concepts of polar coordinates by drawing some curves of polar equations. For example, they will find the following properties about polar coordinates by drawing the curves of r=a, r=bt, r=–bt and by changing the values of "a", "b" and the domain of "t".

1) The value of "t" indicates the measure of angle from the polar axis. Radians are used to express the measure of "t".
2) The value of "r" indicates the distance from the pole to the point in the plane.
3) If "r" is positive, the distance is measured along the ray of angle "t" emanating from the pole; if "r" is negative, the distance is |r| and measured along the ray directly opposite the ray of angle "t".

Students may draw the curve of r=a+bsin(t) and change the values of "a", "b" and the domain of "t" to confirm their findings.

Discussion of Professors Sakitani and Iida's paper:

Demana: We are now open for discussion/comments/questions.

Teague: This last software is an example of the kind of software I was referring to in the last talk. In terms of giving students access to the software over the course of their school career, the other software did one problem and after you've done that problem, after you've rolled the rectangle, it's no good for any other problem. In this software, there's no problem, no particular problem that it's designed for but will handle problems as they come up. And what I'm interested in is what happens to the student in terms of their mathematical thinking if they have access, if the student has access to software like this over their school career?

Sakitani: Well, they don't have much experience of using this software yet, but according to the teacher who created this software, he said that his students enjoyed this activity very much.

Fey: In defense of the drill and practice type program that Professor Iida showed, I looked at some data in the report of Henry Becker (provided by Jerry Becker) about U.S. use of computers and the most popular use by math teachers is drill and practice programs, the second most popular use is programming, and the third is tutorials. What I would like people to comment on is why is it that teachers seem to want one thing from software and those of us who are developing new ways of thinking about the curriculum want something very different, and how can we begin to bring those two points of view closer together?

Iida: At present in Japan, drill and practice software is not very popular to use because this goes along with the present curriculum and it supports the present curriculum very well. Therefore, teachers tend to use this type of software and because there are good sales, the manufacturers try produce a lot more of this type of software. However, when we consider what the computer can do and then certainly the things we're discussing about the problem with the conjectures, discovery, properties, that type of problem will come out.

Fey: Is it really only because drill and practice fits the curriculum. Can we just change the curriculum and have all these other things? I think there's something deeper, a number of things deeper, in teachers' attachment to drill and practice software. I don't know, maybe not. I don't think it's just a matter of changing the curriculum.

Sugiyama: Acquiring the mathematical skills, addition, subtraction, multiplication, division, that kind of skill is very important. By using this type of software, you can change the level of problems suitable for the student and they can drill well and that probably is the main reason for using this type. Another one is that, actually, not very many other types of software are available at present.

Morimoto: The problem solving type of software is not necessarily better than drill and practice type of software. Both are important and, especially, for acquiring the mathematical skills for addition and subtraction types. They require a lot of repetition. The time for each student to achieve the goals is quite different and therefore this type of software is very appropriate because they can give different levels of practice and you can spend as much time as you want on it. The problem right now is that most software available is not much different for the type of problem you give and can solve by using pencil and paper. That is not appropriate, and software should be much more flexible because otherwise there's no really good reason for using computers.

Damarin: It seems to me that one of the issues is that we're switching to a highly visual medium and, if we look at the data from both countries yesterday, what students tended to do was to take visually presented or figural problems and solve them in words, arithmetical expressions, or algebraic expressions. They're much more comfortable transforming a figural problem to something else and then working on it. Most of the software that we've seen here has been highly visual in the way problems are presented and worked on and, I guess, I think we need to work on some transitional kinds of software. The last two pieces that Mr. Iida presented, to my way of thinking, fall into that category as do some other things like green globs, of what I would see as transitional software that many, many kids will need to use in order to really be able to interpret what they see on screens in this more sophisticated software. So, I guess I am talking to teachers - that's my best shot at it.

H. Wilson: Concerning the last presentation, the graphics, I was very impressed with the front-end friendliness of the computer inter-face, in which you were able to make adjustments in the parameters concerning the graphing and it looked very nice. My concern is the speed with which the graph was produced. I'm thinking of the development of the function values over the domain of the function to see it. My own experience in teaching is that as the student watches this, I try to slow it down, rather than speed it up, for understanding purposes. Is there a variable within the software that permits you to slow the graphing function down?

Sakitani: Iida's explanation is the since he's now using the thirty-two bit computer, that's one reason it goes fast. You can probably use a sixteen bit one and slow it down. Also, the language used here is Pascal. If you use Basic or some other language it may slow down too.

H. Wilson: But can you adjust it? Can it be adjusted within the software? No?

Sakitani: No.

Zilliox: I'd like to get back to that issue of the use of drill and practice by teachers more than the use the graphics -type software. To me, in this country, the problem rests more with the teacher feeling comfortable with where the lesson is going to go. With the drill and practice software, as a teacher, I know exactly what a child can do and cannot do and how far they can go with it. I know that it fits the lesson that I have planned for today. To me teachers are threatened by the fact that they no longer have control of where the mathematics is going to go when you have graphics-kind of software. Students can come up with situations that the teacher is unfamiliar with. Every time I play with one of these graphics programs I come up with graphs I don't understand, I don't know where they came from, and as soon as I understand one, someone gives me another that I do not understand. So I think that is uncomfortable for many teachers in their perception of their role of what they're supposed to be doing and what control they have in the classroom, and that's mainly the issue as I see it in this country.

Sakitani: It is certainly true that when I acquired this software it took one whole night to learn it but I really enjoyed it.

Uetake: I would like to give a comment. Number one is about the speed of the program in the computer. Well, certainly it is possible to program it so that the you can slow down the whole process and, in fact, there is some software available which deliberately slow down the whole thing for the students, so that they have enough time to understand. Number two, why is the drill and practice type of software popular? It's because, from the machine point of view, the computer has a very good ability to show on the monitor all the graphics and stuff like that, but what of it's communicative ability? So you input and the computer understands what you input. This ability is not very high yet and that's why it is difficult to make sophisticated software, especially like a tutorial type of software, which requires a lot of response recognition from the student and then further response to it in order to proceed. Especially in the case of tutorial type of software, it requires not only understanding the student's response but to that response the computer has to respond accordingly. If it's the drill and practice type, the decision made by the computer is just "yes" or "no" and so this is quite simple; but in case of the tutorial system, there are a lot of very delicate responses you have to determine and recently, by using artificial intelligence, this type of software is being developed but not yet perfected.

Morimoto: Number one, in responding to Professor Zilliox, it's not really moving from a drill and practice type to the problem solving type or doing something new. They are entirely different types of software and should be used on different occasions or for different reasons. The drill and practice type of software should be used after the students learn how to operate and then by using drill and practice, the purpose is to fix that ability into the students; whereas the problem solving type is to promote student thinking ability. Therefore, they are entirely different. And, in responding to Professor Uetake's opinion about the tutorial type of software, the responses seem quite simple, but actually they are not because, depending on how the student answers the question, the computer has to respond and give different types of the problems or responses; therefore, this type also requires artificial intelligence to improve the software.

Sakitani: Sugiyama is not looking down on the drill and practice type of software. The reason he spends so much time on this software for graphing is because he just wanted to show that with help of computers, the student can really acquire what is said to "do math" or "explore math." He wanted to show the importance of that and really did that.

Demana: Our time has run out. Let's once again join in thanking our presenters for an excellent presentation.

End of Discussion

Discussion of Working Group - Software Demonstrations (JAPAN):

Becker: Now that we've seen a number of very interesting software demonstrations, let's begin this session. The purpose of this session is to comment on, raise questions and discuss the software that we've seen demonstrated by our Japanese colleagues.

Choate: First, can I come teach in Japan? More seriously, I am very impressed with the quality of the software I've seen. I think you have some wonderful pieces of software and I wish we had some of it in the U.S. Particularly, I'm impressed with the simulation software. I have the feeling that much of the software that is developed in the U.S. is very cleverly written, but some of the mathematical content isn't there. I'm thinking specifically now of the piece that I saw demonstrated with the circles and decomposing circle and pieces and obviously a lot of thought went into that and I compliment whoever was responsible for it.

Becker: At the beginning of the demonstration at this machine, we saw some software that was coordinated with one or more of the textbooks. I wonder if our Japanese colleagues could tell us to what extent this occurs in other or all of the textbooks that are used in Japanese schools; that is, is the software commonly coordinated with the mathematical content in the textbooks?

Sugiyama: Not every textbook has software attached to it. For instance, the software we demonstrated here in this room was made either last year or the year before and they are now in the process of making software for the elementary school level and eventually they will come up with more software packages.

Becker: And is the software is coordinated with the textbook.

Sugiyama: Yes, it is.

Sawada: Not only textbook companies, but many computer manufacturing companies and software manufacturing companies are making educational software. In many cases they sell it with the computer as a set. In most cases, the software is arranged so that it will be adaptable to any of the textbooks, not just one.

Morimoto: Compared with the number of software packages for elementary and middle schools, there aren't very many available for the high school level yet. For one thing, it's harder to make. But there is some software related to graphs that is available - right now there are about five of them available.

Uetake: In many cases, high school teachers are not satisfied with software made by somebody else.

Morimoto: Well, what Professor Uetake says used to be true, that the high school math teachers used to make their own materials. But now they are beginning to understand it's quite hard to do that and so many of them are changing to use commercial software, rather than the ones they made for their own use.

Sawada: In my research on software use, not only for math, but any kind of software, I found that in the elementary schools about 76% of the software was commercially made, in the case of the middle schools about 66%, and in the high schools, about 74%. The rest is made by either individual teachers or groups of teachers. Further, when teachers make or create their own software, it is considered private and their own, so when they retire, that's the end. Further, when a teacher develops one software package, that's it and he/she may not desire to make any more.

Morimoto: The recent trend is that the there seems to be a lot of the communication between the personal computer users' group and especially, in each prefecture, there are the educational centers and around the educational centers there is a lot of communication and teachers exchange their own programs or software. Now, regarding what Mr. Sawada said, i.e., once a teacher makes one software and finds it's very hard and, so, maybe doesn't want to make any more, this may be true, especially when it is of the tutorial type, because that is very difficult. However, if a teacher tries to make a software or program just for one class unit, then it isn't considered that hard and they try to do more.

Demana: How do your students utilize the computers outside of the classroom? Is this common, and are there activities that might require it?

Morimoto: Using the computer?

Dugdale: Yes.

Becker: Or calculator technology.

Miwa: Mr. Sawada has mentioned technology use in the juku. Do you know about the juku?

Becker: Yes.

Miwa: Then we need not explain.

Dugdale: What did he say about the juku?

Sawada: There are many jukus in Japan that are adopting computers and software is used there, outside the regular schools. And, most of the software used at the juku are of a drill and practice type.

Choate: I'd like to get back to the subject of the teachers and the software. I think a major problem is who creates the software. Having been in this development field for about the last ten years I think that the Japanese are, not in terms of quality but in terms of the process, probably about four years behind where we are in terms of how good software is produced. Some of the best software that we have in our country is produced by teachers and sort of makes the rounds through what's called share ware or public domain software. And I hope that you don't lose that. And I don't know if you have a method for letting it be known who's written good software, but I highly recommend that you do that. I'll be glad to show anyone who'd like to see some collection of stuff that I've just gathered that teachers have produced that aren't big programs but they do little things wonderfully. It reminds me a lot of some of the demonstration and simulation stuff that I've seen that you've written. And you have them all collected in one place. We don't have that, but they're there. And I hope that you don't discourage teachers from creating software because they're the people that are going to be working with it and I think that you're going to find that they're a wonderful source of ideas. The process of developing software, the one that I think that works and is probably the best, is to come up with a prototype and play with it. Teachers will often make the prototype. They may not polish it but they'll get the initial seed and the initial idea. I would hope that you would give some thought to how do you support a teacher who comes up with a good idea. Do the publishers do that? Do the universities do it? So I would not forget the teachers.

Morimoto: Teachers are beginning to use the modum a lot so that through the telephone cable they exchange software ideas and problems.

Kaji: The software we've demonstrated today is all closed and teachers cannot do anything more with it. However, in the new types of software available, you purchase it and if you know how to program, you can make changes so that you can adapt it to your own class or use you have in mind. And this trend is beginning to grow a lot.

Miwa: In Japan, is there any association or group which provides opportunities to share or exchange software?

Uetake: Once the Ministry of Education and Ministry of International Trade and Industry tried to organize such an association, but it didn't work out. But right now, many software companies are trying to establish such associations. The Ministry of Education tried to evaluate and control the available software, but many software companies are opposed to it because the Ministry of Education is responsible for evaluating textbooks, but software is not included. They don't want the software to be included.

Sugiyama: It seems that the teachers do not appreciate the intervention of the Ministry of Education and so, for example, in case of the books which are required for exercise, they have a kind of independent association, independent from the Ministry of Education, that evaluates books for teachers. Right now the movement is trying to form the same kind of association for evaluating the software outside the Ministry of Education.

Sawada: Well, about three years ago the National Institute of Educational Research (NIER) was asked to organize this kind of evaluation system, but it declined because it was too expensive and requires a lot of people to work on it. But a new academic information center (or something like that) has been established with the main purpose to gather software made by individuals and evaluate it, and then try to sell it at very reasonable prices.

Uetake: Well, for the manufacturers, the educational field or schools are not a good market.

Becker: I would like to come back to the reference to the group of people that evaluates books for the school, and now the group that would evaluate software for the schools (Sugiyama's remark). Who are the people who serve as evaluators? Are they teachers, or parents, or professors?

Sugiyama: As far as evaluation of software is concerned, it's not in process right now - it's not working yet. As for the books which are required for exercise, if it's for elementary schools, then elementary school teachers do the evaluation; and for the middle schools, middle school teachers are involved and maybe one or two college professors as advisors.

Nohda: In Japan, software for educational purposes is not very highly evaluated or appreciated. If it is for engineering, then they are very highly evaluated and thought well of. How about in the states? Oh, let me clarify. It's not the software that's not highly evaluated, rather the job of making software is not highly regarded within the mathematics field. Making good educational software is not a highly regarded occupation whereas it might be in an engineering or information technology field. And if anyone knows about this situation in the U.S., I'd like to hear about it.

Demana: There are some attempts at the university level, for example, recently to have software evaluated for purposes of promotion and tenure. A little bit of that's been going on, not a great deal, and it depends on where you're at in the university level; for example, certain schools would reward very highly production of software, writing textbooks, but others don't. It's a mixed situation in our country. I think the more prestigious the university, the less likely they are to reward that kind of activity, but there is some.

Miwa: Thank you very much. Probably there are still more comments and questions, but, unfortunately, our time is up, so we have to finish this discussion. I thank you for your kind cooperation.

End of Discussion of Working Group

WHERE DO FUNCTIONS COME FROM?
DATA ANALYSIS IN SECONDARY MATHEMATICS

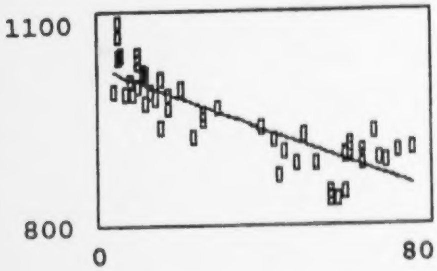
Dan Teague

North Carolina School of Science and Mathematics

> The depth of coffee in an urn being emptied at a constant rate is given by $D = (5.2 - .0082t)^2$ where t is measured in seconds and D in centimeters. What is the depth after 135 seconds. When will the urn be empty?
>
> The temperature of a hot cup of coffee cooling on a table is given by $T = 180e^{-.012t} + 76$, where t is measured in minutes and T in degrees Fahrenheit. When will the temperature be 105 degrees?
>
> The average score on the SAT test (S) is a linear function of the percentage (P) of students in a state taking the test. If $S = -2P + 1015$, what score would you predict for a state with half of its students taking the test? If the average score for a state with 65% of its students taking the test is 900, is it doing well or badly on the test?

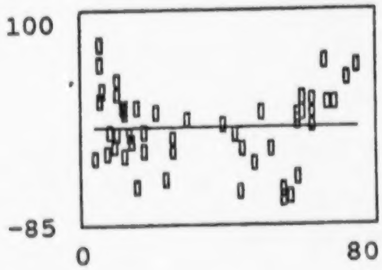
Questions similar to these appear in all secondary mathematics textbooks. In each question, the students are given a function which describes a relationship between variables of interest. From where do such functions come? How are they determined? What mathematics is needed to derive these equations, and what technology is required to assist in their discovery?

The secondary mathematics curriculum focuses primarily on the algebra of the elementary functions. Students learn to graph and manipulate equations involving linear, quadratic, power, exponential, logarithmic, and trigonometric functions. Their study is often accentuated with applications, in "real world" situations similar to the examples above, of the functions and techniques under consideration. In these applications, the students are given the functional form describing the relationship involved in the application. Secondary students in mathematics rarely have the opportunity to use their knowledge to develop functions that describe the world around them. Questions such as: How does the time needed to fill a coffee cup depend upon the depth of the coffee in the urn? What function best describes the relationship between the temperature of the coffee over time if it is left out to cool? How much does the state of North Carolina need to improve its SAT scores, given the percentage of students taking the test? How have the educational programs affected the spread of AIDS in North Carolina? What is the best group size to use when pooling blood samples to test for the presence or absence of a certain characteristic? How can the maximum population be predicted for a logistic growth model? All of the questions, and many others similar in nature and depth, are appropriate for study by secondary mathematics students. The mathematical tool which must be added to the present curriculum is the tool of data analysis. The computational tool required is a computer and graphical data analysis software, such as a spreadsheet. Adding these tools to the curriculum dramatically alters the questions it is feasible to consider and the manner in which students use their knowledge and understanding of mathematics. By using their mathematical knowledge and some techniques of data analysis, high school students can see the interplay between the mathematics they study and the phenomena being modeled by that mathematics in many different areas of human endeavor.



$SAT = -2.11P + 1015$

Figure 1

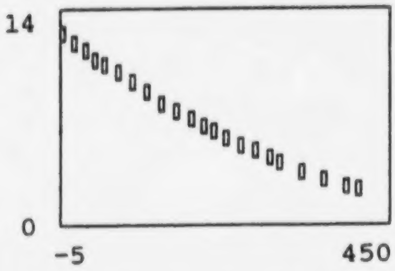


Residual Plot

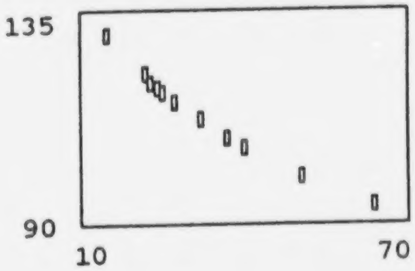
Figure 2

Figure 1 shows a data set from the Raleigh, North Carolina newspaper, *The News and Observer*. It illustrates the average Scholastic Aptitude Test (SAT) score by state in 1990 plotted against the percentage of students taking the test. What information is stored in this data? Which states are doing well? Is a state with an average score of 1000 doing better than a state with an average score of 950? The data looks reasonably linear. By fitting a regression line to the data with a graphing calculator, students see that the relationship $S = -2.11P + 1015$ gives a description of the general trend in the data. What does the slope of the regression line represent? The effect of increasing the percent of the student population taking the SAT by one unit is an expected drop of approximately 2.11 points on the state average score.

Figure 2 is another plot which sheds significant light on a set of data. It is called the residual plot. The residual plot is the set of ordered pairs (x,y – fit), and represents the errors between the fitted y-values and and the actual data. By looking at the residual plot, however, we can see that a state which has an average score of 950 with 65% of its students taking the test is actually doing better in this regard than the state with an average score of 1000 with only 12% of its students taking the test. The residual plot in this case removes the effect of the percent taking the test. These two plots effectively turn the data into information. As we shall see, analysis of the residuals tells us much about the data and about our fit.



Coffee Depth vs Time
Figure 3

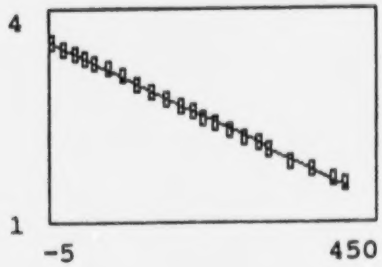


Coffee Temperature vs Time
Figure 4

Students experience phenomena that can be described by the elementary functions every day. Each of them has at one time or another filled a cup from an urn. They can quickly tell whether the urn is nearly empty or full, without looking inside. If they are in line to get a cup of punch, the experience of the first in line is much different from that of the last. The first fills his cup much more rapidly than the last. What is the relationship between the depth of fluid in the urn, and the time that it takes to fill a cup (Figure 3)? It is a linear function only if the experiences of the first and last are the same. If not linear, then what? Another example comes from the cooling of a cup of coffee (Figure 4). Compare the two graphs. Do they represent the similar functional relationships? Does the coffee cool as a quadratic, or exponential, or inverse-square? How can we get information about the nature of these phenomenon? Such everyday experience can be modeled nicely with the functions students study in secondary school, and students observe and experience them every week.

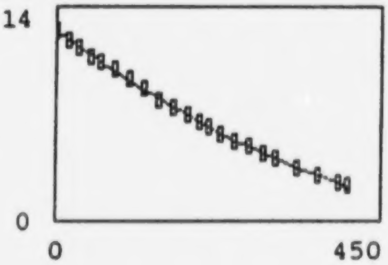
Consider first the process of filling the cup. If this phenomenon represents quadratic behavior, how could we tell? Could it be represented by $D = at^2$. Why or why not? How about $D = a\,t^2 + b$? Or perhaps $D = a\,(t + b)^2$? If we argue that $D = a\,(t + b)^2$ is a reasonable guess, how could we verify it? How is it possible to approximate the values of the parameters a and b ? If $D = a\,(t + b)^2$, then what is the character of the graph of $\sqrt{D}$ against t ? If we were to graph $(T, \sqrt{D})$ rather than $(t\,,D)$, what would we expect to see? Figure 5 describes the square root re-expression. The linearity of the re-expression argues for quadratic behavior of the phenomenon.

The line $Y = -.005t + 3.52$ with $Y = \sqrt{D}$ describes the relationship between time since opening the value and the depth of the coffee in the urn. The model to use then is $D = (-.005t + 3.52)^2$. Graphing this function against the original data shows an excellent fit (Figure 6).



$\sqrt{D}$ vs t

Figure 5

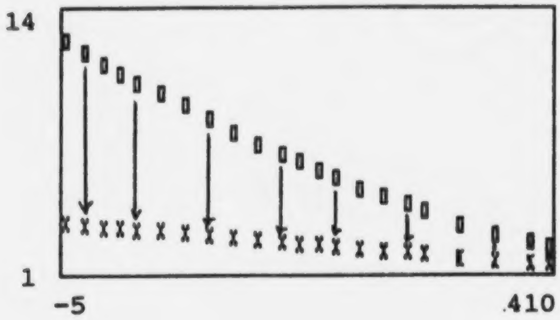


$D = (3.52 - .005t)^2$

Figure 6

How does this re-expression work? By graphing the square root of D, the vertical scale of the graph is altered. Each D-value greater than 1 is pulled down, but the larger values of D are altered more than the smaller D-values. The square root re-expression pulls down more on the large values than on the small ones. This action tends to straighten out the curve.

Such linearization of data is nothing more than an application of the ordinary composition of functions. In every algebra class, students are taught that $f^{-1}(f(x)) = x$, over the domain of f. In the case of linearizing data, what is desired is a kind of "pseudo-inverse", a function g where $g(f(x))$ is reasonably linear. That is, is it possible to find a function g so that $g(f(x)) = mx+b$?



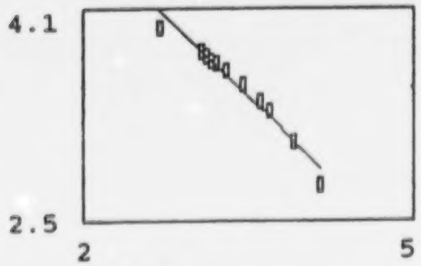
Square Root Re-expression to Linearize Data

Figure 7

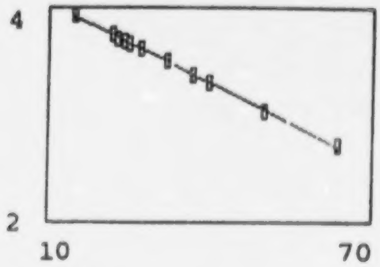
Once g is known, (in this example $g(x) = \sqrt{x}$, it is then possible to solve for f, the function representing the phenomenon of interest. If $g(f(x)) = mx+b$, then $f(x) = g^{-1}(mx+b)$. It is then a simple matter to graph $f(x) = g^{-1}(mx+b)$ against the original data and consider the residuals to determine the quality of the fit. This application of the composition of functions is very powerful and is an excellent motivation for learning to work with both compositions and inverses of functions. It is also possible to re-express the independent variable so that the re-expression $f(g(x)) = m \cdot g(x)+b$ is linear. At times, as with log-log re-expressions, both the dependent and independent variables are re-expressed to linearize the data.

Now look at the cooling problem from Figure 4. Like the previous data set, the graph is decreasing and concave up. Is the cooling of the coffee also an example of quadratic behavior? This example differs from the previous in that there is clearly a vertical shift as well as a horizontal shift. Since the coffee will cool only to the temperature of the room in which it sits, the vertical shift represents the ambient temperature, in this instance 78°. Could cooling be represented by $T= a(t-b)^2 + 78$? If so, then $(t, \sqrt{T-78})$ should be reasonably linear. Graphing this re-expression shows that it is not linear, which argues that the phenomenon isn't quadratic in the form given. Could this decreasing curve be represented by $T = \frac{a}{t^2}+78$, the inverse square relation. Consider the equation $(T-78) = \frac{a}{t^2}$. Taking the natural logarithm of both sides gives $\ln(T-78) = \ln\left(\frac{a}{t^2}\right)$ which implies that $\ln(T-78) = \ln(a) - 2\ln(t)$. The graph of the data set $(\ln(t), \ln(T-78))$ should then be linear with a slope of -2. Notice that all power functions $y=ax^n$ can be reduced through this log-log procedure to lines with a slope of n and a y-intercept of $\ln(a)$. Figure 8 illustrates this log-log re-expression. The phenomenon doesn't appear to be behaving as a power function. If it is not a power, could it be an example of exponential decay?

If the relationship is exponential, then $T= Ae^{-kt} +78$. Subtracting 78 from both sides gives the equation $(T-78) = Ae^{-kt}$. The equation can be rewritten by taking the natural logarithm of both sides. This operation yields $\ln(T-78) = \ln(Ae^{-kt})$ which generates the "linear" equation $\ln(T-78) = -kt + \ln(A)$. Therefore, the graph of $(t, \ln(T-78))$ should be linear with a slope of $-k$. This semi-log graph is shown in Figure 9. As this re-expression appears reasonably linear, the cooling process can be described as an exponential phenomenon.



ln(*T*–78) vs ln(*t*)
Figure 8



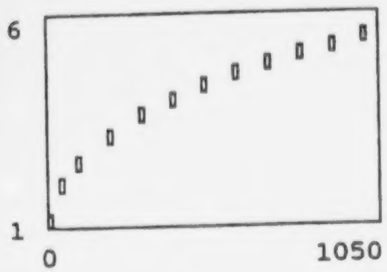
ln(*T*–78) = –.024*t* + 4.293
Figure 9

The fitted line is $Y = -.024X + 4.293$ where $Y = \ln(T-78)$ and $X = t$. This implies that $\ln(T-78) = -.024t + 4.293$. Exponentiating both sides and solving for *T* generates the expression $T - 78 = (e^{-.024t})(e^{4.293}) \rightarrow T \doteq 73.3e^{-.024t} + 78$ as our model. It appears that the temperature of the hot water has a decay rate of approximately 2.5% per minute and a terminal temperature of approximately 78° F.

In all of the data sets seen so far there was quite a lot of variability in the response variable. When introducing students to data analysis and the techniques of re-expression, there is a great need for interesting, yet clean data sets with a high ratio of signal to noise. Such pristine data sets help the students to see clearly the effect of simple re-expressions and aid in the interpretation of the information about that transformation given by the residuals. The greater the variability of the data, the more difficult for the beginner to judge the effects of the transformations. Coupled with this is the desire to give students interesting and challenging "real world" problems. The introduction of graphing calculators into the secondary curriculum offers data sets with the desired characteristics.

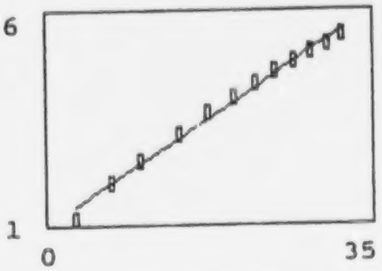
An exceptionally good source of data comes from the approximate solutions to the traditional max-min problems generated with computer and calculator tools. Consider the standard problem of finding the minimum surface area of a right circular cylinder of fixed volume, say 314 cc. The equation needed, $A = 2\pi r^2 + \frac{2V}{r}$, is readily derived by secondary algebra students. The students can then approximate the minimum value by employing a spreadsheet to create a table of values or using a graphing calculator. Either way, an approximate solution of $r \doteq 3.68$ is quickly found. The right circular cylinder with a volume of 314 cc which has the minimum surface area has a radius of approximately 3.68 cm. Typically, the problem ends here. However, the investigation can be extended. If the volume were doubled to 628 cc, would the radius also double? For each specific volume, a particular radius will minimize the surface area. Clearly, the radius which minimizes the surface area is a function of the given volume. But what function? What is the relationship between volume and radius which minimizes the surface area?

To find out, assign each student one or two specific volumes to find the radius which minimizes the surface area using the approximating techniques appropriate for graphing calculators. The assigned volume and the resulting solution for the radius becomes the data set (V, r). What does this data set look like? (See Figure 10)



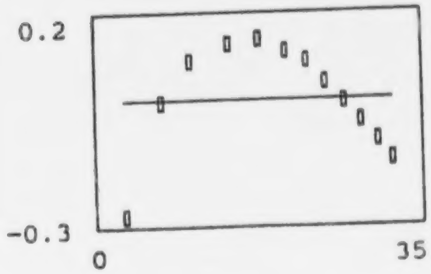
r vs V

Figure 10



r vs $\sqrt{V}$

Figure 11



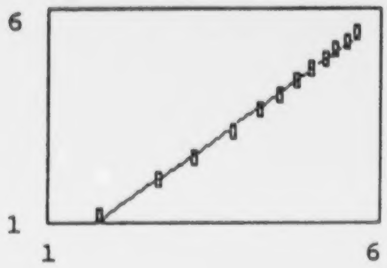
Residual Plot

Figure 12

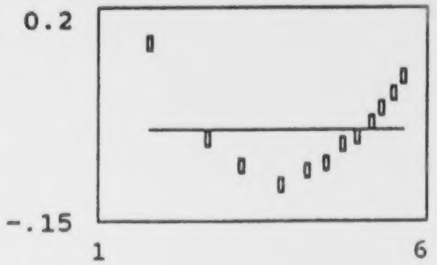
The question before the students is, "Do you know a function which looks like this?" Students will generally guess a square root function, $r = a\sqrt{V}$. If the function is, in fact, a member of the square root family, then graphing either the ordered pairs (V, r^2) or $(\sqrt{V}, r)$ should linearize the data. Below is the re-expression $(\sqrt{V}, r)$ chosen by my class (Figure 11). The transformed data clearly has less curvature and the correlation coefficient is a healthy .991, but is the data linear?

An analysis of the residuals yields a great deal of information about the fit. The residual plot (Figure 12) shows a group of data points below the line $y = 0$, another group above, and a third group again below. This pattern in the residuals is the result of a line being drawn through a curved set of data. A line will partition a curve into three distinct sections. The pattern created by this partition will give evidence to the nature and direction of the curvature. Detecting such patterns is the key to linearizing data. This residual plot indicates curvature which is concave down in the re-expressed data. As that was the concavity in the original data set, the re-expression as a square root is not sufficiently strong to linearize the data. It is necessary to try a re-expression which is stronger than the square root. One such re-expression is the fourth root.

If the function is a fourth root function, $r = a\,V^{1/4}$, then graphing either $(V^{1/4}, r)$ or (V, r^4) should produce a linear graph (Figure 13).



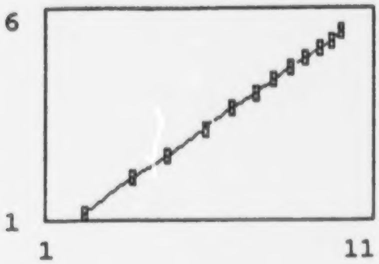
r vs $V^{1/4}$
Figure 13



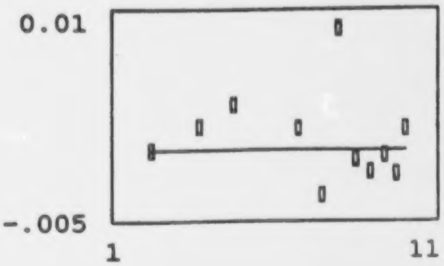
Residual Plot
Figure 14

Although the correlation coefficient for the re-expression ($V^{1/4}$, r) is .997, the residuals (Figure 14) indicate a poor fit from the re-expression. However, this re-expression has altered the concavity of the data. The residuals demonstrate this clearly.

The 1/2 power was not strong enough, leaving the transformed data concave up; but the 1/4 power is too strong, creating a transformed data set that is concave down. The linear re-expression, then, must lie somewhere between the two. The cube root is the most obvious next choice. (Figures 15 & 16)



$r = .524V^{1/3} + .003$
Figure 15



Residual Plot
Figure 16

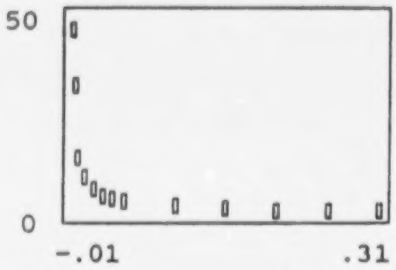
The graph of the re-expression ($V^{1/3}$, r) appears quite linear. Since the residuals support this linearization, the equation $r \doteq .542V^{1/3} + .003$ can be used as our initial model, although we need to reconsider the y-intercept of .003. It should be noted that the correlation coefficient for this model is .999, only marginally better than the previous value of .997. However, comparing residual plots for the cube root model and the fourth root model gives dramatic evidence for the cube root model.

One must always decide what to do with the y-intercept generated by the fit, is it real or a residue of the fitting process? Arguing that zero volume should result in a zero radius, we conclude that the radius-intercept of $.003 \to 0$ and our model for the relationship between V and r is the simpler power function $r = .542V^{1/3}$. It is interesting to note that, from calculus, we find that if $A = 2\pi r^2+2V/r$, and $dA/dr = 0$, then $r = (V/2\pi)^{1/3}$. As expected, then $(1/2\pi)^{1/3} \doteq .5419$.

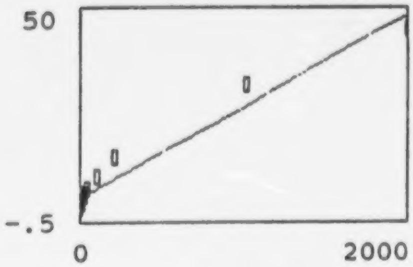
Another perhaps even more powerful example comes from a problem in testing blood for the presence or absence of a particular trait (Meyer). Suppose that you have a large population that you wish to test for a certain characteristic in their blood. Each test will be either positive or negative. If the blood could be pooled by putting a portion of, say, ten samples together and then testing the pooled sample, the number of tests could be reduced. If the pooled sample is negative, then all the individuals in the pool are negative, and we have checked ten people with one test. If, however, the pooled sample is positive, we know only that at least one of the individuals in the sample will test positive. Each member of the sample must then be retested individually and a total of 11 tests will be necessary to do the job. The larger the group size, the more we can eliminate with one test, but the more likely the group is to test positive. The larger the value of p, the smaller the group size should be. Ignoring any Diophantine aspects to the problem, the relationship between the number of tests (T), the size of the population (N), the probability of testing positive for each individual (p), and the group size (k) is given by $T= N/k + (N/k)(1-(1-p)^k)k = N(1/k + (1-(1-p)^k)$. Considering T as a function of k with fixed parameters N and p, it is interesting to consider the relationship between p and k which minimizes T. From calculus $dT/dk = N(-1/k^2-(1-p)^k \ln(1-p))$. If $dT/dk = 0$, then $-1/k^2 = \ln(1-p)\cdot(1-p)^k$. The desired variable k is both algebraic and transcendental in this equation. An approximate solution to this problem can be derived at the precalculus level by utilizing the technology of the graphing calculator and techniques of data analysis. By using the calculator to approximate the value of k which minimizes T for various values of p, a table similar to that below can be constructed:

p	.3	.25	.2	.15	.10	.05	.04	.03	.02	.01	.005	.001	.0005
k	2.7	2.8	2.9	3.2	3.8	5	5.6	6.3	7.6	10.5	14.9	32	45

Look at the graph in Figure 17. As expected, as p approaches zero, the size of the group increases rapidly. Students can argue for a vertical asymptote at $p = 0$. Could the relationship between p and k which minimizes T be a simple reciprocal function?

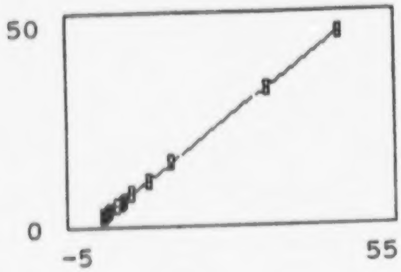


k vs p
Figure 17

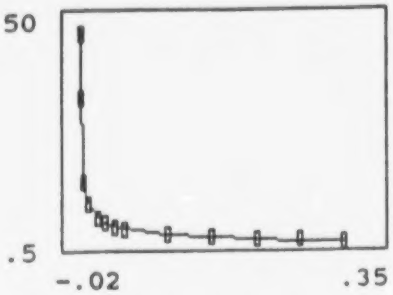


k vs $1/p$
Figure 18

The graph of $(1/p, k)$ (Figure 18) has a different curvature from the original data set. What is needed is some form of reciprocal function which is not as strong as $1/p$. Consider the re-expression $(1/\sqrt{p}, k)$. From the graph of the re-expressed data (Figure 19), it is clear that this re-expression linearizes the data. We can use $k = 1/\sqrt{p} + 0.7$ as our model relating the probability p to the group size k. This model is graphed against the data in Figure 20.



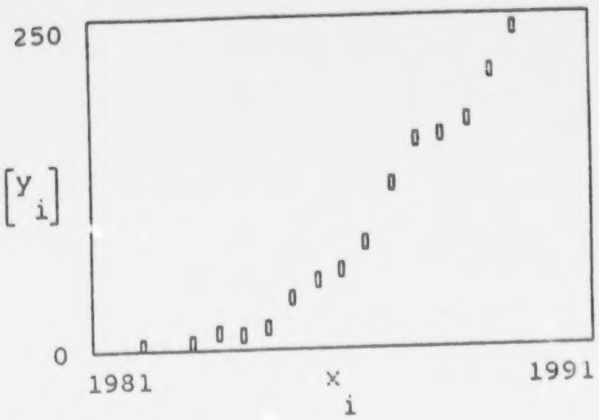
k vs $1/\sqrt{p}$
Figure 19



$k = 1/\sqrt{p} + .7$
Figure 20

One problem from the precalculus final exam concerns the following data set. It shows the number of reported cases of AIDS in North Carolina every six months since 1982.

$x_i :=$	$y_i :=$
1982	4
1983	5
1983.5	12
1984	11
1984.5	17
1985	39
1985.5	52
1986	59
1986.5	79
1987	123
1987.5	156
1988	158
1988.5	169
1989	206
1989.5	239



Data from the North Carolina Department of Public Health

It is easy to show that the first four years of the epidemic can be reasonably described by exponential growth, while the last four years are fairly linear. This is a good example of a piece-wise phenomenon. The characteristics of the disease seem to have been altered by the educational programs begun in 1984 which seem to have altered the characteristics of the growth of the disease by 1986.

In a final precalculus example, the table below illustrates a portion of a wind chill table from the World Almanac. The table is used to determine the wind chill index as a function of the velocity of the wind in miles per hour and the temperature measured in degrees Fahrenheit. By fixing the velocity of the wind, students fit linear models to the rows comparing the actual outside temperature and the wind chill temperature. For a wind velocity of:

5 mph $C \doteq 1.058T - 4.650$

10 mph $C \doteq 1.164T - 22.86$

15 mph $C \doteq 1.343T - 31.34$

20 mph $C \doteq 1.425T - 38.46$

25 mph $C \doteq 1.425T - 38.46$

30 mph $C \doteq 1.521T - 48.10$

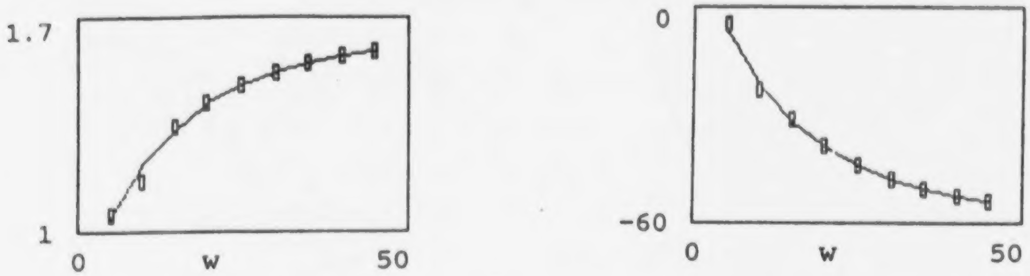
35 mph $C \doteq 1.551T - 50.71$

40 mph $C \doteq 1.573T - 52.49$

45 mph $C \doteq 1.586T - 53.76$

Wind in mph	Temperatures in degrees Fahrenheit																
	35	30	25	20	15	10	5	0	-5	-10	-15	-20	-25	-30	-35	-40	-45
5	33	27	21	16	12	7	0	-5	-10	-15	-21	-26	-31	-36	-42	-47	-52
10	22	16	10	3	-3	-9	-15	-22	-27	-34	-40	-46	-52	-58	-64	-71	-77
15	16	9	2	-5	-11	-18	-25	-31	-38	-45	-51	-58	-65	-72	-78	-85	-92
20	12	4	-3	-10	-17	-24	-31	-39	-46	-53	-60	-67	-74	-81	-88	-95	-103
25	8	1	-7	-15	-22	-29	-36	-44	-51	-59	-66	-74	-81	-88	-96	-103	-110
30	6	-2	-10	-18	-25	-33	-41	-49	-56	-64	-71	-79	-86	-93	-101	-109	-116
35	4	-4	-12	-20	-27	-35	-43	-52	-58	-67	-74	-82	-89	-97	-105	-113	-120
40	3	-5	-13	-21	-29	-37	-45	-53	-60	-69	-76	-84	-92	-100	-107	-115	-123
45	2	-6	-14	-22	-30	-38	-46	-54	-62	-70	-78	-85	-93	-102	-109	-117	-125

By fitting the linear models, the students created two new data sets, with the independent variable being the wind velocity and dependent variable the slope and *y*-intercept of the regression line.



Slope vs Wind

Intercept vs Wind

Figure 22

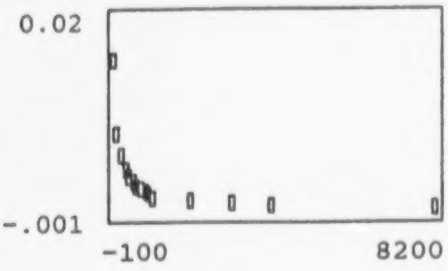
Fitting these data gives a multivariate prediction model of wind chill temperature as a function of both wind velocity and actual temperature of

$$C(w, T) \doteq (1.623 - .829e^{-.07w})(T) + (70.316e^{-.066w} - 57.5).$$

As described above, data analysis provides a wonderful motivation for and application of the topics in precalculus. But what happens when the student enters the calculus class? Once the student enters the calculus class, data analysis takes on an even greater role. The students can investigate all of the data sets generated in precalculus from a new perspective. In precalculus, students learned that functions can be defined by its data set. In calculus, they learn that functions can also be defined by how the data set changes. To study the changes in the data, a new data set is derived from the original. To derive the new set, replace each y-value with the local average rate of change of the function. That is, replace y_i with $\frac{y_{i+1}-y_{i-1}}{x_{i+1}-x_{i-1}}$. This value is called the symmetric difference. Since the first and last data points do not have values on both sides, the original n data points generate a derived set of $n-2$ data points. When the derived data sets are compared to the original data sets, something exceptional happens. Every quadratic data set collected in precalculus generates a linear derived data set. Every exponential data set generates another exponential data set. The derived set is characteristic of the original data set. It doesn't take students long to decide that if they know what the derived set is, through the usual re-expression techniques of data analysis, then they also know, up to a constant, what the original data set is.

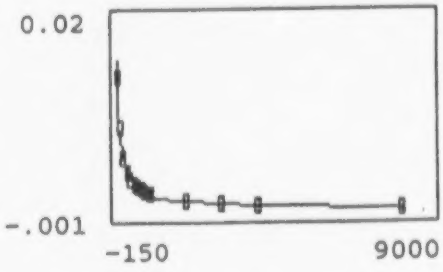
For example, the data set which came from minimizing the surface area of the can. Our precalculus analysis determined the relationship to be $r = .542V^{1/3}$. If we generate the derived set, we have the data set shown in Figure 23. By re-expressing the data, we see that the derived data set is a negative two-thirds power function. The average change in a cube root function seems to be a negative two-thirds power function! Furthermore, the coefficient of the new function is approximately 1/3 of that of the original. Such observations are an excellent way to motivate the basic differentiation formulae.

Similar results can be generated from all of the data sets in the precalculus course. After generating these approximate results and making conjectures about the relationship between the original data set and the derived set, the conjectures can be verified with the development of the traditional differential calculus.



$\frac{\Delta r}{\Delta V}$ vs V

Figure 23

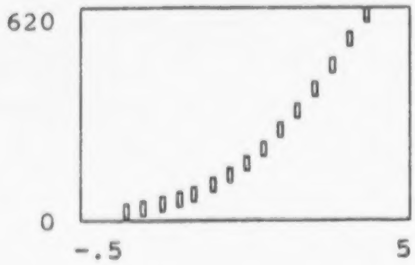


$\frac{\Delta r}{\Delta V} = .18V^{-6.7}$

Figure 24

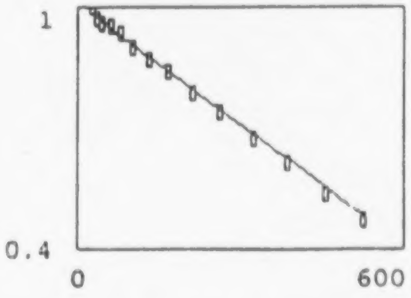
One of the data sets previously considered from precalculus is that of hot coffee cooling on the table. By re-expression the data with $(t, \ln(T-C))$ and considering this semi-log graph, it was argued that the cooling process could be reasonably modeled by $T = Ae^{-kt} + C$. This, of course, is known as Newton's Law of Cooling. However, in order to obtain this result at the precalculus level, it was necessary to have a good estimate for C. In the previous example, the coffee was left in a room for several hours to allow for a good estimate of C. The semi-log re-expression is useless if the student cannot remove the constant prior to using logarithms. In the calculus course, such extensive data collection is not necessary. In the derived set, such constants are irrelevant. The derived set for the cooling data is a negative exponential function which is asymptotic to the x-axis, and is easily linearized. The initial condition gives back the value of C.

As a further example of this, consider the logistic curve $P = \frac{M}{(1+Be^{-Mkt})}$. In this particular case, let $M = 1000$, $B = 39$, and $k = .001$. Such a function can be linearized with the re-expression $(t, \ln(\frac{M-P}{P}))$. However, the value of M must be known to accomplish this linearization. Suppose, however, that the data at hand is the portion of the logistic curve seen in Figure 25 and the goal of the data analysis is to estimate M. The logistic curve is also defined by the differential equation $dP/dt = kP(M-P)$. This means that the graph of the symmetric differences, approximating d P/dt, plotted against P should be quadratic. Furthermore, the graph the ratio of the symmetric differences to P, $\frac{dP/dt}{P}$, graphed against P should be linear, since $\frac{dP/dt}{P} = kM - kP$.
If we generate this re-expression, we should be able to approximate both k by considering the slope, and then M by considering the y-intercept. This re-expression is shown in Figure 26.



P vs t

Figure 25



$\frac{\Delta P/\Delta t}{P} = -.00103P + 1.00939$

Figure 26

Notice that the slope of -0.00103 implies that $k = -.001$ and the y-intercept $1.00939 = kM$ implies that $M = 1000$.

It should be clear from the examples given above that computer tools are essential for utilizing the power of data analysis in the curriculum. To be proficient at data analysis as described here, students must have firmly in their mind the shapes of the elementary functions and how translations, reflections, and compositions affect them. They achieve this familiarity by seeing computer generated graphs every day in class and graphs generated by graphing calculators in the evening. Students are constantly sketching graphs by hand to illustrate their qualitative behavior. Access to these graphing tools is essential.

In addition to the graphing tools, data analysis tools must be available. This need not be a specialized data analysis software package, any spreadsheet that allows the graphing of data will suffice. Although the graphing calculators will perform the data analysis, they are presently too cumbersome for extensive work in data analysis.

Data analysis adds tremendously to the student's understanding of functions. It offers the students a tool with which to approach the unknown world around them, but it requires the computational aid of a computer to successfully re-express and linearize the data. By combining the techniques of data analysis and the power of graphical software tools, with the traditional study of secondary algebra, students will see very clearly where functions come from!

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Discussion of Mr. Teague's paper:

Sugiyama: Thank you Mr. Teague for your very interesting presentation. Now I will open the discussion session.

Hashimoto: I found your presentation very interesting, especially the data and using the graphing calculator to solve the problems was very interesting. Now, there are two points I'd like to ask: (1) concerns the function to find the relationship in the varying data. It seems like you as a teacher already have the given relationship between this and that. What changes is according to what is given, and the varying data are already given. Would it be possible for the students to find the relationship, not just the relationship between two given data sets, but the data themselves? (2) concerns the problem given by the teacher, so the teacher found the interesting phenomenon and then gave this as a problem to the students. Now, is there any possibility, from the students' side, of finding some interesting phenomenon or problems from their daily life and presenting them?

Teague: Yes, the problems I presented, the data that I showed you, were data that students collected. We asked the question about getting coffee from the urn, for students it's punch from the urn. And they all have recognized that if you're there last it takes longer to fill your cup. They've never thought about that in terms of linear functions or non-linear functions, so they haven't tried to describe the phenomenon , so I phrased the question, how could we decide what it is? Some of them decide to take the data set that I used, which is depth versus time, they just open it and measure how deep it is as time goes on. Others take a cup and measure how much time, how long it takes to fill one cup, two cups, three cups, and so they would have the inverse function from the one I showed. But either way, it's the students' decision as to what they want to measure. You're right. I posed the phenomenon but the goal of the course, one goal of the course, is for students to be able to pose their own questions. And the example of the wind chill table is an example of that. That's a problem a student brought in. They found the table in the Almanac and they asked can we find out what's going on, or how can we describe these numbers? So, early in the course, the teacher poses the questions because the students are just learning. Later in the course, we hope the students will pose their own questions and answer them.

Uetake: When talking about this relationship between the temperature and the depth of the urn, who gave that data?

Teague: That was data that the students collected. For homework I had them each go and put a thermometer in the hot water heater and watch it for an hour, and then bring that data in.

Choate: At my school we use the materials that have been developed in Mr. Teague's school and we've done this experiment a lot with students. A question that naturally arises is does it make any difference what kind of cup you have? Why do the fast food companies package things differently? And we've actually taken field trips where I take an entire class down to Dunkin Donuts and we do experiments. I think the point is that the mathematics gets out of the classroom and the students enjoy it. And we've found that using these experiments that the students suddenly begin to realize that this is not just something that you do in a classroom but there are things you can find out using this method and we've had great luck with this experimental approach of collecting data and trying to find patterns.

J. Wilson: Dan, I understand the point that it's the students' problem. The students are gathering the data. But, it's your problem. To what extent do you have problems where the students have really formed the problem? Hey, let's organize and gather a bunch of data on this? That's one question. The second question is that it's easy to believe that this works with Dan Teague. I wonder what experience you've had with how well it works with the whatever this concept of "average teacher" means fifty miles down the road?

Teague: The students fairly often will come up with problems on their own. Again, it's more a phenomenon later in the course once they get comfortable with looking at data, but often times it's an extension of something that we've done rather than something that's totally new. I had a group of students compare McDonald's, Burger King's and Hardee's coffee cups by looking at slips of similog lines and sort of taking five of each and averaging the rate of heat loss. I've had them filling up a balloon with air and let it go to examine the time that it stays in the air as a function of the circumference which tends to be cubic, as you might suspect. Measuring the temperature inside a car with the windows closed in the summer. That tends to be exponential. But those are examples of problems students just did and brought into class to share. About regular folks, we have had our course used in a lot of different areas, but the course is the kind that is used generally by teachers like Jon Choate who are very good and very free at taking risks. And so we don't have much experience with teachers that would, we have experience with schools that are more normal but not teachers that are more normal.

Demana: Dan, I know you have to worry about evaluation of students. All teachers do, and they tend to also generally ask some regular questions. Would you speak to how your students do on what one might call ordinary kinds of questions having had this kind of approach?

Teague: They are generally bored with ordinary kinds of questions. They like the exploration. Not all students, some students like to remember mathematics rather than think mathematically. But in large part our students enjoy the challenge, particularly when there's a question like the wind chill question when I haven't a clue, they have, no one knows and the goal is to see what we can say. And it's really a difference in the end product of knowing the mathematics and being able to do it and an end product of being able to use the mathematics. And our goal is that second one, to be able to use it.

Damarin: Dan, it seems that a lot of your problems demand that students engage in indirect reasoning unlike some of the other examples that we've had where they have to deal with if relationship A holds, then B holds; not B, therefore not A. And the data, the research data tell us that's pretty challenging; it tells us that that's often difficult for many students and I wonder if you have to teach that or if kids just catch on to that way of reasoning after multiple exposures?

Teague: It is hard for students but it's something that we do consistently from the first because I think it's important. It's an important skill for students to have, to think, when you approach a problem you look at what you know, how does what you know fit what you've got? You make a conjecture. And then you say if I'm right, what should happen? And does it happen? It's also important to realize that sometimes you don't know. At the end of the process you still don't know. That it is not something that we can do. And I think that it's important for students to realize that in not every data set you can find some sort of functional, simple functional relationship. But it is something we have to teach. And you work at it and again as I said, it's much more teacher-directed in terms of how you, how would you progress at this, what would be your first thought? Your second thought? Towards the end goal, in the paper I gave a set of the AIDS data for North Carolina and that set of data was on the final exam and the question is what do you see? What would you do with this set of data? How would you approach it? They don't have the tools to do it, but what I want to know is what, you know, what would you do? What would you expect to see? What would you be surprised to see? Often that's the best question. What would really shock me?

H. Wilson: I'd like to get back a little bit closer to what Frank Demana asked earlier. If we have a national curriculum in the United States I think an example of it would be the AP calculus syllabus. How does this fit in that context and whether or not these students have any success therein? I assume it's a calculus course that they would take for the AP calculus.

Teague: We have been forced not to teach AP calculus as a result of teaching precalculus this way. We have changed, next year we will teach no AP calculus whatsoever because the difference in the courses was so dramatic. No teacher wanted to teach calculus. There were students who wanted to take AP calculus but by and large the students, it's a very different kind of thinking and it's almost like once you open the gate and allow students to put part of themselves into a problem, it's very difficult to then close the gate and say you simply have to remember there are rules to learn and rules to apply. And we don't know how that's going to pan out in the long run, but we will not, we won't teach an AP calculus. Students can still take it and we will be interested in how well they do, but we don't know that yet.

Kaida: I think you'd better explain what AP calculus is.

Teague: AP calculus is a standard college calculus curriculum so that a student who takes calculus in high school can receive college credit, university credit by taking and doing well on the AP test.

Becker: Is there time for one more question? Thank you. Dan, I missed the very first part of your talk, for which I apologize. And maybe you spoke to this, but I'm interested in the logistics of how these problems unfold. Does the posing or the formulation of the problem happen pretty much with you and the whole class together, and then the students are left to pursue it individually or in small groups as projects? How does that occur?

Teague: That varies again as we go through the course. If I give you the first data set in the first week and ask you what should we do, who knows, no one knows what to do. And so early on I give examples of what we could think, what do we know, how could we use what we know. And so it's early on more of a demonstration, taking students' suggestions, but often times students' suggestions don't come early on. Later it's more, as you said, here's a situation, someone has brought in something, or I brought in something. Is there anything interesting in it? Or it may be more directed, try to find a certain relationship and then they, we do a lot of group work, groups of three, where they talk among themselves and decide an approach, to say well it could be rational or it could be exponential. And if it's rational we should do this, if it's exponential we should do that. What do we do first, what's the more likely? And then we come together and try to find out what people have found out, what people thought were interesting.

Becker: Do you think we should stop the discussion for a break now?

Teague: We can talk at the break, or is there another question?

Kaida: The Japanese also use this kind of approach, the giving of the data and finding whether it's on a curve or a straight line and then find out the equation for that. The similar thing is done in the study of statistics as well, so do you find any relationship between them and how do you relate that?

Teague: The process is the same, the focus for us is on the function and the interaction of functions and so our point of view in the process is as a motivational tool for helping students learn about and wanting to, developing an interest in learning about functions.

Sugiyama: Thank you. We have no time left, so we need to close the discussion.

End of Discussion

THE USE OF COMPUTERS IN SCHOOLS: SOME OF THE FINDINGS FROM NATIONAL AND INTERNATIONAL SURVEYS

Toshio Sawada
National Institute for Educational Research
Tokyo, Japan

Introduction

The extent of the use of computers in education in Japan, particularly in elementary and secondary schools, has been rather limited, although experimental projects on the use of computers for CAI/CAL in schools have been undertaken for more than 20 years mainly by researchers and professors at universities. The low use rate is partly due to the fact that up to now in the official school curriculum, (i.e., the Courses of Study issued by the Ministry of Education), computers have not been treated as a part of the teaching content except for the curriculum of vocational upper secondary schools. So only a limited number of schools have been using and giving instruction about computers.

With the rapid development and extensive use of computers in society and in business, in particular, a need has come to be felt to introduce computers into education and to give instruction about computers as well as to make use of them for other than administrative purposes. Currently, an increasing number of schools are being equipped with microcomputers and they are trying to make use of computers for the improvement of instruction and to teach students about computers. Various surveys including the one coordinated by the International Association for the Evaluation of Educational Achievement (IEA) have been conducted to find out the present situation and trends in terms of the use of computers in elementary, lower and upper secondary schools. Some of the results will be discussed in the following sections. The data quoted below without named sources are from the results of the national surveys conducted by the Ministry of Education, Science and Culture (1991).

1. Some of the Results from National Survey

1-1. Availability of Computers in Schools

How many computers do schools possess? As shown in Table 1, according to the national surveys conducted in all Japanese public schools by the Ministry of Education, Science and Culture, 0.6% of elementary schools, 3.1% of lower secondary schools and 56.4% of upper secondary schools had more than one microcomputer in 1983. In 1990, these figures increased to 30.9%, 58.9% and 97.8%, respectively. This increase was achieved partly because of the measures taken by the Ministry of Education to facilitate the introduction of computers in view of the emerging need to prepare students for the forthcoming information-oriented society. The Ministry started to subsidize the purchase of computers by public schools in 1985. Subsequently, in line with this, local governments have also been encouraging the introduction and use of computers in schools through their subsidy systems.

Table 1 Diffusion Rate of Computers in Japanese Public Schools

	Elementary Schools	Lower Secondary Schools	Upper Secondary Schools
May 1, 1983	0.6%	3.1%	56.4%
Oct. 1, 1985	2.0	12.8	81.1
March 31, 1987	6.5	22.8	86.3
March 31,1988	13.5 (3,337)	35.6 (3,748)	93.7 (3,925)
March 31, 1989	21.0 (5,172)	44.8 (4,740)	96.3 (4,035)
March 31, 1990	30.9 (7,600)	58.9 (6,229)	97.7 (4,090)

Note: The figures in parentheses are the number of schools with computers.

But as shown in Table 2, the introduction of computers into schools has not progressed to the extent it deserves, especially in terms of the number of computers installed in each elementary and lower secondary school. As of March 1990, the average number of computers in those schools that have computers is 3.1 computers per school for elementary schools, and 5.5 and 29.8 for lower and upper secondary schools, respectively.

Table 2 Average Number of Computers per School

	Elementary Schools	Lower Secondary Schools	Upper secondary Schools
March 31, 1988	2.9 (9,523)	3.5 (13,199)	19.7 (77,420)
March 31, 1989	3.0 (15,505)	4.3 (20,519)	25.5 (103,014)
March 31, 1990	3.1 (23,572)	5.5 (34,069)	29.8 (121,900)

Note: The figures in parentheses are the number of computers installed in schools.

About seventy-five percent of computers installed in schools have been purchased by schools with the help of subsidies from both national and local government as shown in Table 3.

Table 3 Methods of Procurement of Computers (Year: 1990 and 1989)

	Elementary Schools		Lower Secondary Schools		Upper Secondary Schools	
	1990	1989	1990	1989	1990	1989
Purchase	77.4%	81.4%	80.1%	83.8%	79.9%	80.4%
Rental/Lease	16.3	11.4	15.0	9.2	17.9	17.1
Others (1)	6.3	7.2	4.9	7.0	2.2	2.6

Note: (1) Donations/ gifts, etc.

As of 1990, about 30% of computers installed in elementary schools are 8-bit machines, and 13% and 18% of those in lower and upper secondary schools, respectively, are also 8-bit machines. However, the majority of computers installed in schools are 16-bit machines and there is an increasing tendency to install more 16-bit or 32-bit machines in schools.

Table 4 Type of Computers in Schools (Year: 1990 and 1989)

	Elementary Schools		Lower Secondary Schools		Upper Secondary Schools	
	1990	1989	1990	1989	1990	1989
8-bit machine	31.1%	39.9%	12.8%	19.3%	17.8%	22.5%
16-bit machine	66.2	58.3	85.5	79.3	78.6	75.5
32-bit machine	2.0	1.3	1.4	1.0	2.7	1.1
Others	0.6	0.5	0.3	0.5	0.9	0.9
Total number of computers in schools	(23,572)	(15,505)	(34,069)	(20,519)	(121,900)	(103,014)

Regarding the location of computers as shown in Table 5 , the survey showed that many schools keep their computers in the faculty rooms or teachers' rooms. However, there is a growing tendency to keep computers in computer laboratories, in special rooms reserved for the teaching of certain subjects, or in multi-purpose or audio-visual rooms. One of the major reasons for keeping computers in teachers' rooms, apart from the security factor, is that many schools which introduce computers for the first time tend to use them for administrative or management purposes in the beginning due to the availability of the number of computers as well as the number of teachers who can operate computers in the schools.

Table 5 Location of Computers in Schools (Year: 1990 and 1989)

	Elementary Schools		Lower Sec. Schools		Upper Sec. Schools	
	1990	1989	1990	1989	1990	1989
Computer Laboratories	11.3%	9.4%	19.6%	16.5%	55.4%	49.3%
Special Rooms (1)	9.4	10.9	11.1	12.2	51.2	48.2
Classrooms	5.0	5.9	1.7	1.6	2.6	2.3
Libraries	3.7	3.6	2.2	2.6	8.6	6.9
Teachers' Rooms	69.3	66.4	63.9	63.0	59.3	55.1
Others (2)	18.2	19.5	28.4	29.9	43.6	41.4

Note: Computers are kept in more than one place, so the total percentage exceeds 100%.

(1) Such as special rooms for the teaching of science, industrial arts and home making, and audio-visual rooms.

(2) Such as multi-purpose rooms and in-school broadcasting studios.

1-2. Software Development and Use

Which software is available in the schools? The availability of instructional software is another necessary condition for using computers in schools. Therefore, this survey contained a number of questions about the availability of software in the schools.

The average number of software items possessed by public elementary schools in 1990 was 24.4, and 37.8 and 105.4 for public lower and upper secondary schools respectively. The number of software possessed by schools has greatly increased from the previous year as shown in Table 6. However, many schools have multiple copies of the same software item, and the number of different kinds of software, excluding multiple copies of the same kind of software, possessed by schools is 8.4, 9.0 and 20.2 per school for elementary, lower secondary and upper secondary schools, respectively.

Table 6 Number of Software Items Possessed by Schools
(Year: 1990 and 1989)

	Elementary Schools		Lower Sec. Schools		Upper Sec. Schools	
	1990	1989	1990	1989	1990	1989
Average Number of software items per schools	24.4	19.6	37.8	27.5	105.4	78.5
No. of schools with computers	7,600	5,172	6,229	4,740	4,090	4,035
No. of software items processed	185,241	101,623	235,538	130,306	430,948	316,768

The results of the software items possessed by schools are presented in Table 7. Of the software items possessed by schools in 1990, 76.1%, 66.2% and 74.2% of software were commercial software items purchased by elementary, lower secondary and upper secondary schools, respectively. The remaining software was either developed by school teachers or jointly developed by teachers and local education centers.

About 68%, 66% and 57% of software possessed by elementary, lower secondary and upper secondary schools, respectively, are for use in connection with subject teaching, whereas the remaining software are for administrative use, etc..

Table 7 Sources / Development of Software used in Schools
(Year: 1990 and 1989)

	Elementary Schools		Lower Sec. Schools		Upper Sec. Schools	
	1990	1989	1990	1989	1990	1989
Teacher made	13.5%	15.0%	10.3%	23.1%	19.3%	21.4%
Joint development (1)	5.8	6.4	20.1	14.6	2.7	2.5
Commercial	76.1	73.9	66.2	59.7	74.2	68.9
Others (2)	4.6	4.8	3.2	2.5	3.8	7.2

Note: (1) Developed jointly by teachers and staff of local education centers, and by groups of teachers.
(2) Exchange with other schools, gifts, etc.

Besides the software for use across subjects, as shown in Table 8, software for Mathematics, Japanese Language and Science is popular in elementary schools, software for Mathematics, Science and Foreign Languages (English) in lower secondary schools, and software for Vocational Subjects, Mathematics and Science in upper secondary schools.

One of the interesting features of Japanese education is the development and the existence of an education industry. Textbooks are developed by private publishers, although they have to be approved by the Ministry of education. Apart from textbooks, drills, worksheets, test and supplementary instructional materials are also developed by private publishers and used in schools, and are also available in the market for home use.

Table 8 Software for Subject Teaching in Schools (by Subject)
(Year: 1990)

Subjects	Elementary Schools	Lower Sec. Schools	Upper Sec. Schools
Japanese Language	10.5%	3.6%	1.1%
Social Studies	6.4	4.9	0.8
Mathematics	52.3	25.9	7.6
Science	7.7	24.4	6.2
Music	2.7	0.6	0.2
Fine Arts	2.4	1.2	0.2
Industrial Arts	-	6.0	-
Homemaking	0.1	0.9	1.1
Physical Education	0.6	0.6	0.3
Foreign Languages	0.1	13.0	1.8
Moral Education	0.04	0.1	0.02
Special Activities (1)	2.4	1.6	0.9
Vocational Subjects	0.0	0.0	49.9
Across subjects (2)	14.8	17.3	29.9

Note: (1) Such as for club activities.
(2) Word Processing; Spreadsheets; Graphics; Database; etc..

An increasing quantity of commercial software is now available in the market, but evaluation of this is not yet systematically carried out. Although general guidelines for the development of software have been issued by the Ministry of Education in order to ensure the high quality of software, the selection of or method adopted for the evaluation of software is up to the user.

Some information on the content and evaluation results of new software is disseminated through monthly magazines and journals published by private publishers. A semi- governmental organization established in 1986 is engaged in collecting and evaluating software developed by teachers and schools as well as that used in pilot/experimental projects, and in making the results of the analysis available to users.

1-3. Curriculum

Why is it difficult to integrate computers in the school curriculum? The Courses of Study for elementary, lower secondary and upper secondary schools have been revised almost every ten years, and the most recent revision has just taken place. New Courses of Study will be put into practice from 1992 for elementary schools, from 1993 for lower secondary schools and from 1994 for upper secondary schools.

In the new Courses of Study, computers are expected to be more positively used than now in the teaching of various subjects, and computer education will be covered by such subjects as industrial arts and homemaking in the lower secondary schools, as well as such subjects as mathematics and science in both lower and upper secondary schools, as well as vocational subjects in vocational and technical upper secondary schools.

According to the new Course of Study for elementary schools, it is expected that use will be made of computers for the improvement of teaching and learning, and that through the use of computers as an aid children will make themselves familiar with computers. But it is not expected that the functions and operation of computers will be taught at this level.

In the Course of Study for lower secondary schools, it is expected that use will be made of particular characteristics and functions of computers such as simulation and information retrieval for instruction and that through such use of computers students' understanding of computers will be deepened and necessary skills will be developed. In the subject " Industrial arts and Homemaking ", there will be a requirement to teach the content of the foundation of information, especially computers, to both boys and girls.

In the Course of Study for upper secondary schools, in the teaching of each subject, proper attention has to be given to the emergence and development of an information-oriented society and the impact of computers on individuals and society. In mathematics teaching, a new subject area called ' Mathematics C " will be introduced to teach students about computers focusing on computer use. The only computer-related section in the present curriculum is found in terms of computers and flowcharts in the present "Mathematics II". Indeed, if we compare this with the new mathematics curriculum, we see a phenomenal change.

1-4. Teachers

How many teachers can operate computers? A suggested training program has been developed by the Media Committee of the Social Education Council and the Ministry of Education, as well as local (prefectural) governments, have been providing interested teachers with opportunities for in-service education in the use of computers. In particular, prefectural boards of education make use of their education centers or establish information processing education centers to provide training opportunities for teachers. However, in-service teacher education provision varies from prefecture to prefecture. Teachers who wish to undergo training are provided with opportunities for training on the basis of the availability of courses.

Teachers interested in the use of computers may also find their own ways of getting trained at their own expense, as private companies provide teachers with opportunities for training on computers. Topics covered by training courses vary from course to course depending on the specific objectives/purpose and levels (beginner, intermediate, advanced, etc.) of courses, and the levels or backgrounds of participants. A training course normally covers general theory and the operation, programming and application of computers. Pre-service teacher education programs with respect to computers also vary from institution to institution.

As shown in Table 9, according to the survey conducted by the Ministry of Education last year, 10.1%,18.3% and 32.4% of public elementary, lower secondary and upper secondary school teachers, respectively, responded that they can operate computers and 20.3%, 27.6% and 42.8% of those elementary, lower secondary and upper secondary school teachers, respectively, responded that they can teach about computers. Those teachers who are able to operate computers are teachers of mathematics, science, industrial arts and homemaking, in the case of lower secondary schools, and those of vocational subjects, mathematics and science in upper secondary schools.

Table 9 Percentage of Teachers who can operate Computers
(Year: 1990)

	Elementary Schools	Lower Sec. Schools	Upper Sec. Schools
Can operate computers (No. of teachers who can operate computers)	10.1% (44,494)	18.3% (50,294)	32.3 (71,142)
Can teach about computers out of those who can operate them	20.3%	27.6%	42.8%
Japanese Language	7.7%	7.3%	
Social Studies	-	10.0	7.2
Mathematics	-	20.4	17.9
Science	-	22.3	14.6
Music	-	3.4	0.7
Fine Arts	-	3.3	0.5
Industrial Arts & Homemaking	-	14.3	-
Homemaking	-	-	2.0
Physical Education	-	6.8	5.1
Foreign Languages	-	6.5	8.3
Vocational Subjects	-	-	34.4

Table 10 presents the results of forms of training received by school teachers. So far, about 115,800 teachers have undergone some form of training in the use of computers, and a little over half of them (56%) received training organized by national and local government/prefectural education centers.

Table 10 Forms of Training received by Teachers
(Year: 1990)

	Elementary Schools	Lower Sec. Schools	Upper Sec. Subjects Schools
National/Local Governments	58.9%	54.7%	55.1%
University extension courses, etc.	4.6	5.8	4.9
Research Associations, etc.	14.2	15.0	14.6
Manufacturers/companies	15.8	17.9	20.0
Others(1)	6.5	6.6	5.4

Note: (1) Self-study, In-school training, etc.

2. Some of the Findings from IEA Computers in Education Study

2-1. Why are schools using computers?

Many countries are facing issues about the role of computers in education. Do computers have a place in the school? What should be their role? At what educational level should students be introduced to computers? What should they be taught? Who should teach them? In which existing courses can computers be used most effectively as a tool to improve the teaching-learning process? What will be the effects of computers on students? On teachers? On the school as an institution? These are important questions and, at present, we have little information to guide us in answering them. The overall aim of the IEA computers in education study is to contribute to building a knowledge base from which answers to the above questions about what and how to use computers in education can be sought.

In 1989-90, the International Association for the Evaluation of Educational Achievement (IEA) conducted the stage 1 survey of computers in education (COMPED) in the schools of about 20 countries, including the United States and Japan. This preliminary report presents the first results of stage 1 of the COMPED study. Other results will appear in a research volume to be published in 1991, in national reports of the participating countries, and in articles in scientific journals,(e.g. William J. Pelgram & Tjeerd Plomp: the Use of Computers in Education Worldwide; results from the IEA 'Computers in Education" Survey in 19 Education Systems; in printing at Pergamon Press.1991b)

The school principals participating in this study were asked to rate the importance of each statement in a list of nine containing potential reasons for introducing computers in the schools. The answer categories were: not important, slightly important, important and very important.

The list consisted of the following statements: (1) students need experience with computers for their future, (2) computers make school more interesting, (3) computers attract more students to the school, (4) computers improve student achievement, (5) computers keep the curriculum up to date, (6) computers promote individualized learning, (7) computers promote cooperative learning, (8) the school had an opportunity to acquire computers, (9) the teachers were interested.

In order to visualize the major trends with respect to the reasons for introducing computers more clearly, we collapsed the answer categories " important" and "very important" and calculated the percentage of respondents checking one of these answers. Table 11 presents the results of these responses for elementary, lower secondary and upper secondary schools in the United States and Japan.

Table 11 shows that at all levels the expected improvement of student achievement is mentioned by a large majority of respondents. In elementary schools, Japanese principals reported that their important items more than 70% were (1), (2), (4), (5), (6), (7) and (8) , while principals of the United States selected items (1), (4), (5) and (6). In lower secondary schools, the items more than 79% of Japanese principals were items (1), (5), (6), (8) and (9), those of US were items (2), (4), (5) and (6). In upper secondary schools, those reported that their important items were (1), (5) and (8) for Japan, and (1), (4), (5) and (9) for the United States. Both countries gave their relatively lower rating to (3) computers attract more students to the school, and (7) computers promote cooperative learning.

Table 11 Reasons for Introducing Computers in Schools (Principals)

Reasons Introduction	Elementary Schools JPN	Elementary Schools USA	Lower Sec. Schools JPN	Lower Sec. Schools USA	Upper Sec. Schools JPN	Upper Sec. Schools USA
(1) Experience for future	91.%	92.%	88.%	90.%	92.%	94.%
(2) Make school interesting	83.	64.	66.	63.	63.	53.
(3) Attract students	40.	8.	37.	9.	62.	14.
(4) Improve achievement	71.	79.	63.	78.	48.	76.
(5) Curric/method up-to-date	80.	91.	73.	94.	76.	92.
(6) Individualized learning	95.	74.	84.	73.	59.	63.
(7) Cooperative learning	54.	48.	55.	47.	41.	40.
(8) Opportunity to acquire	81.	49.	80.	45.	72.	40.
(9) Teachers were interested	69.	67.	73.	67.	67.	79.

Note: JPN = Japan, USA = the United States of America

2-2. How teachers use computers?

The computer-using teachers in this study were asked to rate the frequency of use of each statement in a list of seven containing the following approaches to using computers for your "selected subjects (mathematics, science and mother tongue) in this class. The answer categories were: never, some weeks, most weeks and every week.

The list consisted of the following approaches: (1) drill: students do practical exercises on the computer, (2) instruction by computer: the software provides the actual instruction, (3) explanation/ demonstration: the teacher explains and/or demonstrates an idea or skills,(4) testing: students take tests by using computer software, (5) enrichment: fast learners get additional instruction/ exercises on the computer, (6) remediation: slow learners get additional instruction/ exercises on the computer, and (7) let students explore concepts on their own. We collapsed the answer categories " most weeks" and "every week" and calculated the percentage of respondents checking one of these answers.

Table 12 presents percentage of teachers in existing subject using particular instructional approaches (mathematics, science and mother tongue teachers). Computers may be used for a wide variety of purposes in a school. In the past, some different topologies have been presented to characterize the major distinctions in the type of use of computers, such as learning with, learning about, and learning through computers. Schools, however, are constrained in their use of computers due to a number of different factors, such as the shortage of hardware and software and the limited availability of teacher time, etc..

Table 12 Percentage of Teachers in Existing Subject using Particular Instructional Approaches (mathematics, science and mother tongue teachers)

	Elementary Schools		Lower Sec. Schools		Upper Sec. Schools	
Approach	JPN	USA	JPN	USA	JPN	USA
(1) Drill, practice	85.%	93.%	77.%	77.%	49.%	47.%
(2) Instruction by computer	18.	77.	27.	59.	40.	56.
(3) Teacher demonstrates	41.	58.	56.	50.	59.	65.
(4) Students tested	12.	29.	14.	29.	15.	18.
(5) Enrichment	31.	71.	16.	64.	14.	54.
(6) Remediation	37.	70.	24.	59.	11.	36.
(7) Students self-explore	34.	60.	27.	54.	39.	50.

Regarding the approaches to using computers, many teachers take approaches such as (1) drill & practice for elementary and lower secondary schools in both countries. On the other hand, in items (2), (4), (5), (6) and (7), the U.S. teachers show a relatively higher percentage than Japanese teachers.

As for the opinions and attitudes of teachers toward computers, it was found from the data of the IEA COMPED study made in early 1989 that teachers are generally in favor of computers and have positive attitudes and opinions about the role of computers in society as well as in education. They tend to consider computers as a valuable tool to improve students' learning and enhance teaching effectiveness, and they are eager to learn more about computers as a teaching aid. However, due to the lack of systematic in-service teacher education programs on computers in the past, most teachers, especially those who are not using computers, consider that they do not know much about computers or their operation.

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THE USE OF COMPUTERS IN SCHOOL: THE STATE OF USING COMPUTERS IN THE MATHEMATICS CLASSROOM IN JAPAN

Koichi Kumagai
Department of Mathematics, Joetsu University of Education
Joetsu, Japan

I. Background and Purposes

Education with computers has been a focus of mathematics education. Whenever new teaching approaches and technology are introduced in education, many problems arise in relation to traditional education. Of course, many problems are now arising in mathematics education with computers, in relation to the purposes of mathematics education, curriculum development, teaching methods, teacher education and so on.

In Japan, some schools have computers and others do not. Many teachers have already used computers in teaching mathematics. Teachers are now confronted with real problems. But we do not have enough information about problems in mathematics teaching with computers in school. For example, how does a teacher use computers? How are computers used for demonstration in a class, for drill and practice, and for problem solving in teaching mathematics? Are there any influences of computers on a student's cognitive process, problem solving process, and so on? What kind of influences are there? Now we can identify such problems in real situations.

Several surveys about education with computers have been conducted in recent years, but most of them examined the general features of education with computers in the schools. Surveys for teaching mathematics in a classroom are rare in Japan. In this report we present the analysis of education with computers from the teacher's point of view. The general features of school education with computers (number of available computers, computer laboratories, and so on) and features of teaching mathematics with computers are reported. Particularly, teaching mathematics with computers, teachers' conceptions, and relationships between them are the focus of this report. The teachers' conception includes the influence of computers on mathematics learning, problem solving, and so on.

II. Method

1. Subjects

A total of 143 teachers from seven Japan prefectures (Aichi, Hiroshima, Ibaraki, Kanagawa, Niigata, Tokyo, Yamanashi) participated in the study during winter 1991. Of the total

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participants, 34 were from the elementary, 67 were from the lower secondary, and 42 were from the upper secondary school. Table 1 shows the ages of participants.

Table 1. Age of participants

	Ages				
school levels	20 ~29	30 ~39	40 ~49	50 ~60	
elementary	10	14	9	1	34 (24%)
lower secondary	10	42	12	3	67 (47%)
upper secondary	2	25	12	3	42 (29%)
total	22 (15%)	81 (57%)	33 (23%)	7 (5%)	143 (100%)

2. Questionnaires

Each teacher was given the questionnaires (see Appendix I) in which questions about facts and conceptions were arranged. The facts survey included environments for computer use in school (hardware, software, colleagues and so on: Q.3, Q.4, Q.5, Q.9, Q.10, Q.11) and mathematics teaching with computers (software in class, usage of computers in mathematics teaching: Q.6, Q.7, Q.8). The conceptions survey included teachers' conceptions about the influence of computers on students' learning and problem solving in mathematics (Q.12).

III. Results

In this section, we start with the results of the fact surveys.

Table 2. User & non-user in each school level

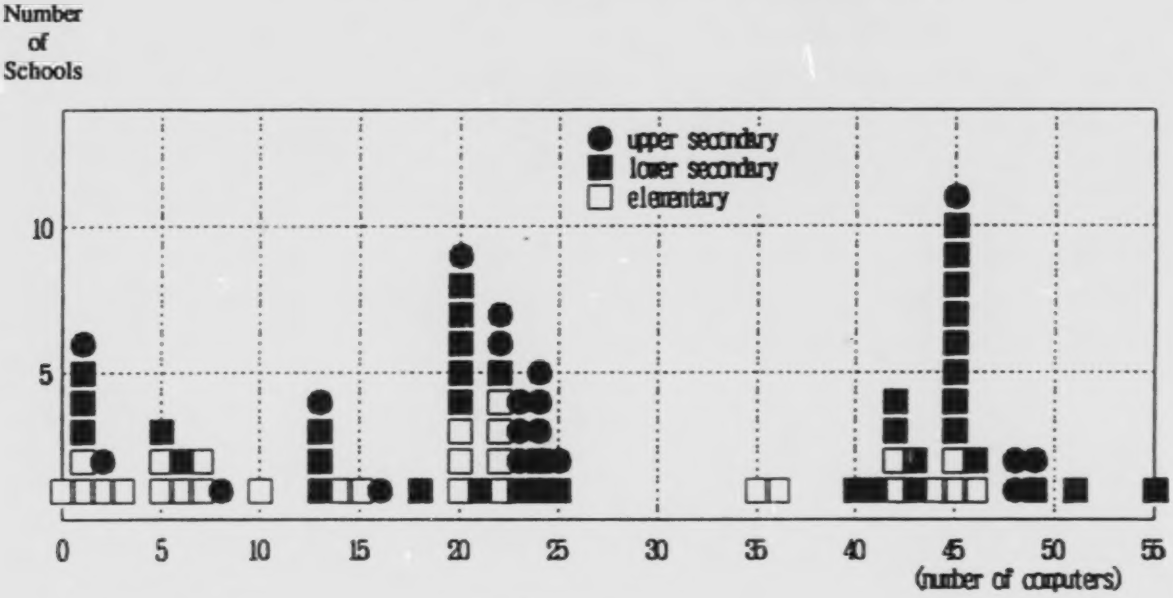
school levels	Yes	No	total
elementary	28 (82%)	6 (18%)	34
lower secondary	39 (59%)	27 (41%)	66
upper secondary	19 (46%)	22 (54%)	41
total	86 (61%)	55 (39%)	141

The total number of responses was 143. There were 86 users of computers, or 60% of the total number of responses. On the other hand, there were 57 non-users, or 40% of the total number of responses (Table 2). ("User" means a teacher who has used computers.)

1. Computer environments in a school

We divided the results of computer environments into three sections: computer environments about hardware, teachers, and teachers' complaints about the state. Most of the result within this section were based on the results from users'.

Table 3. Number of available computers in a school



Hardware

Table 3 shows the distribution of the number of available computers in a school. Table 3 also shows the distribution of the number of available computers in a school. There are more than 40 available computers in 33% of the schools. Most of them are in the lower secondary school. There are 20 available computers in 35% of the schools. 20% of the school have at most 7 available computers. Most computers (78%) are placed in a computer laboratory (see Appendix II, Table 1).

Among teachers

We now consider the environments among teachers. There are two questions.

Q.5 Can you get support from others when you are using a computer in your class?

1. yes (who:) 2. no

Q.9 How frequently are you engaged in each of following activities?

1) talking generally about instructional uses of computers or educational purposes with another teacher
2) talking about professional uses of computers, (e.g. programming, recording grades etc.) with another teacher
3) meeting with teachers from other school to discuss the use of computers

There are three reasons: never, sometimes, and often.

In general 26% of teachers could get support from others when they used computers. In the lower secondary level, teachers could not get much support from others; that is, only 13% (see Appendix II. Table 2).

But most teachers (95%) had opportunities to talk about the educational use of computers. The percentages of teachers who talk about professional uses of computers with each other were 93%, and at least 81% of the teachers had meetings with teachers from other schools to discuss the use of computers. Most teachers actively talked and discussed the proper use of computers for education and teaching, as shown in Table 4.

Table 4. Communication about education with computers

	often	sometimes	never	no answer
educational uses	41 (47%)	41 (47%)	2 (3%)	2 (3%)
professional uses	34 (40%)	46 (53%)	5 (6%)	1 (1%)
have a meeting	18 (17%)	52 (61%)	15 (21%)	1 (1%)

Teachers' reasons concerning computer use

The percentages of reasons for not using computers made by non-users (n=55) is displayed in Table 5. The main reason is that computers have not been available (29%) and that there was not enough software for instructional purposes available (19%). Particularly, in the lower secondary level computers have not been available, and in the upper secondary level there was not enough software. There are not enough computers and appropriate software in schools.

But teachers actively discuss with each other about education and computers.

Table 5. Reasons for not using computers

	elementary	lower sec.	upper sec.	total
insufficient computers	5 (56%)	14 (34%)	7 (18%)	26 (29%)
not enough software	0 (0%)	7 (17%)	10 (26%)	17 (19%)
educational reason	1 (11%)	3 (7%)	6 (15%)	10 (11%)
usages of computers	0 (0%)	2 (5%)	4 (10%)	6 (7%)
not enough time	1 (11%)	3 (7%)	7 (18%)	11 (12%)
lack of experiences	2 (22%)	8 (10%)	1 (3%)	11 (12%)
others	0 (0%)	4 (5%)	4 (10%)	8 (9%)

2. Teaching and Computers

We now consider mathematics teaching and computers. We divided the results of teaching and computers into two sections: software and programs in teaching and usage of computers in mathematics teaching.

Software and Programs

(i) Computer software used in school (Q.6)

Popular types of software which were used in schools were drill and practice programs, word processing/desktop publishing programs, spreadsheet programs, authoring programs for writing CAI lessons, and programs for recording or scoring tests.

Table 6. Software used in a school

	school levels			
types of a software	elem.	lower	higher	total
drill and practice programs	22	25	6	53
tutorial programs	16	24	6	46
word processing/desktop publishing	12	32	11	55
painting and drawing programs	13	24	7	44
educational games	7	9	2	18
simulation programs	7	32	9	48
mathematical graphing programs	4	26	13	43
statistical programs	5	17	4	26
programming languages	6	21	15	42
spreadsheet programs	11	28	12	51
programs for recording or scoring tests for data base	13	33	15	61
database programs	8	23	11	42
authoring programs for writing CAI	15	29	7	51
tele computing	4	2	1	7
	147	326	120	593

Table 6 shows that at the elementary level, drill and practice programs, tutorial programs, and authoring programs for writing CAI lessons are popular. In the lower secondary level, simulation programs were outstanding and word processing/desktop publishing programs and programs for recording or scoring tests were also used. At the upper secondary level, mathematical graphing programs, programs for recording or scoring tests and programming languages were popular.

(ii) Source of programs used in the mathematics teaching

There is a question about programs; that is, "How often do you use these sources of computer programs in your instruction in this mathematics class ?"

(1) programs that I wrote

(2) programs copied from books or magazines

(3) software written with the school system or obtained by exchanges with other schools

(4) other commercial software

In the teaching of mathematics, the majority of teachers used program written by themselves, programs written with school systems or obtained by exchanges with teachers of other schools, and commercial programs, as shown in Table 7.

Table 7. Source of programs used in a teaching mathematics

	original	books etc.	exchange	commercial
many	13 (15%)	4 (5%)	11 (13%)	25 (29%)
several	26 (30%)	5 (6%)	20 (23%)	20 (23%)
once or twice	27 (32%)	22 (26%)	27 (31%)	11 (13%)
nothing	19 (22%)	51 (59%)	26 (30%)	28 (32%)
no answer	1 (1%)	4 (5%)	2 (2%)	2 (2%)

Programs written by teachers themselves and written with school systems or obtained by exchanges with teachers of other schools were used several times or once or twice a year (63%). The commercial programs were used frequently (54%).

Software used in schools is different across school levels. Each school level has characteristics. There are three main sources of software in each school level.

Teaching mathematics with computers

(i) Approaches to using computers for mathematics lessons

There are questions about approaches to using computers for mathematics lessons.

Q.8 How often have you used the following approaches to using computers for your mathematics lessons in this class?

(1) students explore concepts on their own or do practical exercises

(2) demonstration: the teacher demonstrates an idea or skill

(3) testing: students take tests by using computer software

(4) enrichment: fast learners get additional instruction

(5) remediation: slow learners get additional instruction

(6) for students to use as a tool for word processing, calculation, and database

(7) others

The percentages of teachers who use computers to let students explore concepts on their own or do practical exercises is 85%. Teachers had many opportunities to use computers for demonstration and for remediation: slow learners 69% and 40%, respectively. Relatively, they did not use for testing and for enrichment with fast learners (see Table 8).

Table 8. Approaches to using computers in the mathematics teaching

	often	sometimes	never	no answer
1) students explore	25 (29%)	48 (56%)	11 (13%)	2 (2%)
2) demonstration	8 (9%)	50 (58%)	26 (30%)	2 (2%)
3) tests	2 (2%)	17 (20%)	65 (76%)	2 (2%)
4) enrichment	4 (6%)	17 (20%)	63 (73%)	2 (2%)
5) remediation	5 (6%)	29 (34%)	50 (58%)	2 (2%)
6) word processor	10 (12%)	16 (19%)	57 (66%)	3 (4%)
7) others	4 (5%)	0 (0%)	46 (54%)	36 (42%)

The percentages of teachers who use computers to let students explore concepts on their own or do practical exercises was high in every school level, as shown in Table 9.

Table 9. For students explore

	often	sometimes	never	total
elementary	12 (43%)	14 (50%)	2 (7%)	28
lower secondary	8 (22%)	26 (70%)	3 (8%)	37
upper secondary	5 (26%)	8 (42%)	6 (32%)	19

Tables 10 and table 11 show the relationships between the school levels and approaches to using computers for demonstration and for remediation: with slow learners.

Table 10. For demonstration

	often	sometimes	never	total
elementary	1 (4%)	14 (50%)	13 (44%)	28
lower secondary	5 (14%)	26 (70%)	6 (16%)	37
upper secondary	2 (10%)	10 (53%)	7 (17%)	19

Table 11. For remediation

	often	sometimes	never	total
elementary	4 (14%)	6 (21%)	18 (64%)	28
lower secondary	0 (0%)	18 (49%)	19 (51%)	37
upper secondary	1 (5%)	5 (26%)	13 (68%)	19

In the lower secondary level, the percentages of teachers who used computers for demonstration and for remediation: with slow learners were 84% and 51%, respectively.

(ii) The relationships between source of programs and approaches to using computers

Table 12 shows the relationships between source of programs which are used in mathematics classes and approaches to using computers. When teachers let students explore concepts on their own or do practical exercises, they used commercial programs and programs which were written with school systems or obtained by exchanges with teachers of other school. For demonstration, teachers used programs which were written with school systems or obtained by exchanges with teachers of other schools and commercial programs. For remediation: with slow learners, commercial programs were used frequently.

Table 12. The relationships between source of programs and approaches

		Q7			
		1)	2)	3)	4)
Q8	1	5 (6%)	9 (10%)	29 (34%)	43 (50%)
	2	9 (10%)	6 (7%)	23 (27%)	31 (36%)
	3	2 (2%)	4 (5%)	21 (24%)	14 (16%)
	4	9 (10%)	3 (3%)	21 (24%)	15 (17%)
	5	9 (10%)	6 (7%)	18 (21%)	24 (28%)
	6	4 (5%)	5 (6%)	8 (9%)	16 (17%)

3. Influence of teaching with computers on mathematics education

From the viewpoint of problem solving, we now consider teachers' conceptions about the influence of computers on mathematics education. In Q.12, there were 6 questions with three responses: yes, neutral, and no.

(1) By using computers, basic knowledge and skills of mathematics can be mastered.
KS [knowledge and skills]

(2) By using computers, the intuitions of a student can be fostered.
I [intuitions]

(3) By using computers, students' logical thinking can be extended.
LT [logical thinking].

(4) By using computers, students will be able to solve problems which have been unsolved by them previously.
PPS [possibility of p-s]

(5) By introduction of computers, the nature of mathematics problems may change.
CP [changes of problems]

(6) By using computers, we may foster students' problem solving in school mathematics
SPS [student's p-s].

The total number of responses was 143. There were 86 users of computers, or 60% of the total number of responses. On the other hand, there were 57 non-users, or 40% of the total number of responses.

General tendencies

In questions about KS [knowledge and skills], I [intuitions], and SPS [student's p-s], the percentages of positive responses were almost 50%, and that of negative responses were almost 10%. 50% of the teachers responded positively to the question PPS [possibility of p-s] and 20% of the teachers did negatively. 30% responded positively and 15% negatively to the question of LT [logical thinking].

Users and Non-users

We consider teacher's conceptions in comparing users with the non-users. Table 13 shows that the percentage of positive responses made by the users was higher than that of the non-users, except for CP [changes of problems]. The percentage of neutral responses made by the non-users are higher than that of the users, except CP [changes of problems] and PPS [possibility of p-s].

Focusing on negative responses, the negative responses appeared in non-users' responses. Particularly, in the questions of LT [logical thinking], the non-users had negative responses.

Furthermore, examining relationships among 6 questions made by users, there are correlations between KS [knowledge and skills] and I [intuitions], between KS [knowledge and skills] and SPS [student's p-s], and between I [intuitions] and SPS [student's p-s]. There are biases between I [intuitions] and LT [logical thinking], between KS [knowledge and skills] and CC [changes in contents], between KS [knowledge and skills] and PPS [possibility of p-s], and between I [intuitions] and CP [changes of problems].

Table 13. User and non-user in teachers' conceptions

[user]	yes	neutral	no	no answer
knowledge and skills	54 (63%)	28 (33%)	3 (3%)	1 (1%)
intuitions	50 (58%)	28 (33%)	3 (3%)	1 (1%)
logical thinking	33 (39%)	44 (51%)	8 (9%)	1 (1%)
possibility	33 (39%)	37 (43%)	15 (17%)	1 (1%)
changes of problems	38 (45%)	31 (36%)	15 (17%)	1 (1%)
p-s ability	48 (56%)	31 (36%)	6 (7%)	1 (1%)

[non-user]	yes	neutral	no	no answer
knowledge and skills	19 (35%)	28 (50%)	8 (15%)	0 (0%)
intuitions	24 (44%)	25 (43%)	6 (11%)	0 (0%)
logical thinking	12 (22%)	27 (50%)	16 (29%)	0 (0%)
possibility	17 (31%)	24 (44%)	14 (26%)	0 (0%)
changes of problems	31 (56%)	18 (33%)	6 (11%)	0 (0%)
p-s ability	13 (24%)	32 (58%)	10 (18%)	0 (0%)

The percentages of the former is higher than the latter. These results show consistency.

Examining responses of the non-users, there are correlations between KS [knowledge and skills] and SPS [student's p-s], between LT [logical thinking] and PPS [possibility of p-s], and between PPS [possibility of p-s] and SPS [student's p-s]. But there are biases between I [intuitions] and LT [logical thinking], between CP [changes of problems] and LT [logical thinking], between CP [changes of problems] and PPS [possibility of p-s], between CP [changes of problems] and SPS [student's p-s]. In each relationship, the former is more emphasized. There is no consistency with the two results for non-users (see Appendix III, Table 1).

School levels

At the elementary level, negative responses appeared in every question. Particularly, the percentages of negative responses in CP [changes of problems] and PPS [possibility of p-s] made by users are almost 33%. But at the secondary level, positive responses appeared.

In non-users, generally, neutral responses were more than other responses. There were no non-users at the elementary level who had negative opinions. But non-users at the secondary level had negative opinions (see Appendix III, Table 2.-Table 13).

Through these considerations, we conclude that the users have a particular image of the influences of computers, but the non-users have various images.

4. The relationships between teachers' conceptions and their mathematics teaching

We consider the relationships between teachers' conceptions and their mathematics teaching. There were a total of 85 responses from users. At first we examine the relationships between source of programs used in mathematics teaching and teachers' conceptions (see Appendix IV, Table 1). Secondly, the relationship between approaches to using computers in mathematics teaching and teachers' conceptions were examined (see Appendix IV, Table 2).

Source of program and teacher's conceptions

In the previous consideration, there were four following sources of programs:

(1) programs that I wrote: WO
(2) programs copied from books or magazines: CB
(3) software written with the school system or obtained by exchanges with other schools: SE
(4) other commercial software: CS

WO teachers [who wrote programs themselves] passively evaluated influences on LT [logical thinking] and PPS [possibility of p-s]. Negative influences on PPS [possibility of p-s] and CP [changes of problems] appeared from WO teachers.

SE teachers [who use software written with the school system or obtained by exchanges with other schools] positively evaluated influences on KS [knowledge and skills] and I [intuitions]. But the percentages of positive evaluation made by the group of SE is lower than for the group of WO teachers. Negative influences on PPS [possibility of p-s] appeared from SE teachers.

CS teachers [who use commercial software] tend to evaluate positive influence by using computers in general.

We could not find consistency in the teachers' conceptions through the results from source of programs.

Approaches to using computers and teachers' conceptions

There were five approaches to using computers in mathematics teaching:

1.	let students explore concepts on their own or do practical exercises	E
2.	demonstration: the teacher demonstrates an idea or skill	D
3.	testing: students take tests by using computer software	T
4.	enrichment: fast learners get additional instruction	F
5.	remediation: slow learners get additional instruction	S

The overall distribution of results from relationships between approaches to using computers and teachers' conceptions was examined. All teachers positively evaluated influences on

KS [knowledge and skills], I [intuitions], and SPS [student's p-s], and there were some negative responses in PPS [possibility of p-s] and CP [changes of problems]. As a result, teachers' conceptions shows consistency in the results from approaches to using computers.

There is no relation between teaching approach and teacher's conceptions. But we can identify relations between source of programs and teacher's conceptions. That is, we can set such assumptions that teachers who wrote programs themselves have different conceptions from teachers who use commercial programs.

IV. Conclusions

General features

(Environments)

When teachers used computers in the class, they could not get any support from others. But teachers frequently discussed and talked to each other about educational uses of computers and professional uses of computers. Their communication about computers was actively done.

In general, there was an insufficient number of computers available in schools and not enough software available for instructional purposes.

Teaching and computers

Approaches to using computers in mathematics classroom have variety in each school level. Some teachers wrote programs themselves, others used software written with the school system or obtained by exchanges with other schools.

Teachers' conceptions in teaching with computers

(Teachers' conceptions and users and non-users)

In comparing teachers who use computers with those who do not, there are differences in conceptions' influence of computers on mathematics educations. For users, teaching with computers positively influences KS [knowledge and skills], I [intuitions], and SPS [student's problem solving]. But at the elementary school level, some users have negative opinions in PPS [possibility of problem solving] and CP [changes of problems]. On the other hand, non-users have various opinions about influences of computers.

From the results of teachers' conceptions, we find that teachers who have conceptions that "the nature of problems may change" are not so many. This problem related problems of teachers' experiences of problem solving with computers in mathematics.

(Teachers' conception and the mathematics teaching)

The differences among teachers' conceptions depend on differences of source of programs which were used by teachers, and do not depend on their differences of approaches to using computers in mathematics teaching.

In sum, in Japan there are two problems: first, there are problems about environments of computers; second, the problem is teachers' computer literacy. This problem includes not only distinction between user and non-user, but also distinction between writing program oneself and not. Furthermore, teachers' experiences in problem solving with computers in mathematics also should be included.

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[Appendix DI]

Questionaires

Prefecture ____________________ Name of school ____________________

Teacher: Name ______________________ Age ______ Sex male/female

Q.1 Which grade are you teaching now? How often do you use a computer in your classroom?

grade	number	The frequency of using computer in your classroom			
& class	of student	never	sometimes	almost every week	every week
___ ___	______	1	2	3	4
___ ___	______	1	2	3	4
___ ___	______	1	2	3	4

Q.2 Have you been used a computer in your classroom?

1. yes 2. no

If your answer is "no." you may answer the question 11. If your answer is "yes." you may answer the next question 3.

Q.3 How many microcomputers of computer terminals are usually available for useby that mathematics class?

number of computers (at the moment)

number of computers (two years ago)

Q.4 For teaching and learning purposes, where is the computer usually used?

1) in your class
2) in a computer laboratory
3) in other place (precisely:)

Q.5 Can you get support from others when you are using computer in your class?

1. yes (who:) 2. no

Q.6 Which of the following types of software have been used by you at school? Circle one or two usages that apply.

1) drill and practice programs
2) tutorial programs
3) word processing programs/desktop publishing programs
4) painting and drawing programs
5) educational games
6) simulation programs
7) mathematical graphing programs
8) statistical programs
9) programing languages
10) spreadsheet programs
11) programs for recording or scoring tests
12) database programs
13) authoring programs for writing CAI lessons
14) tele computing

Q.7 How often do you use these sources of computer programs in your instruction in this mathematics class?

	not used twice	once or twice	several times a year	more often than that
1) programs that I wrote	1	2	3	4
2) programs copied from books or magazines	1	2	3	4
3) software written with the school system or obtained by exchanges with other schools	1	2	3	4
4) other commercial software	1	2	3	4

Q.8 How often have you used the following approaches to using computer for your mathematics lessons in this class?

	never	sometimes	often
1) let students explore concepts on their own	1	2	3
2) demonstration: the teacher demonstrates an idea or skill	1	2	3
3) testing: students take tests by using computer software	1	2	3
4) enrichment: fast learners get additional instruction	1	2	3
5) remediation: slow learners get additional instruction	1	2	3
6) for students to use as a tool for word processing, calculation, and database	1	2	3
7) others	1	2	3

Q.9 How frequently are you engaged in each of following activities?

	never	sometimes	often
1) talking generally about instructional uses of computers or educational purposes with another teacher	1	2	3
2) talking about professional uses of computers, (e.g. programming, recording grades etc.) with another teacher	1	2	3
3) meeting with teachers from other school to discuss the use of computers	1	2	3

Q.10 To what extent would you use a computer in your classroom or teaching and learning purpose in the future if given the opportunity?

1) willing to use computers regularly

2) not very willing to use computers

3) may use computers if a certain condition is arranged

In question 12, if your answer was "yes," you may skip the next question 11, and answer Q.12.

Q.11 What are the main reasons for not using a computer on teaching and learning purposes in your classroom? Circle one or two reasons that apply.

1) insufficient number of computers available

2) not enough software for instructional purpose available

3) an educational reason and purposes were unclear

4) how to use a computer in the classroom was unclear

5) not enough time to use computers

6) lack of experience using computers

7) others:

Please answer the next question 12.

Q.12 Regarding problem solving activities in a mathematics class, this question involves the role of computer usage.

	yes	neutral	no
1) By using computers, basic knowledge and skills of mathematics can be mastered.	1	2	3
2) By using computers, the intuitions of a student can be fostered.	1	2	3
3) By using computers, students' logical thinking can be extended.	1	2	3
4) By using computers, students will be able to solve problems which have been unsolved by them previously.	1	2	3
5) By introduction of computers, the nature of mathematics problems may change.	1	2	3
6) By using computers, we may foster students' problem solving in school mathematics	1	2	3

Q.13 Write a paragraph on the role of computers regarding problem solving in school mathematics.

[Appendix. II]

Table 1. Places for using computers

school levels	classroom	room for computer	others
elementary	5	19	4
lower secondary	2	34	3
upper secondary	1	14	4
total	8 (9%)	67 (78%)	11 (13%)

Table 2. Supports from others using computers

school levels	yes	no	no answer
elementary	10 (37%)	17 (63%)	1
lower secondary	5 (13%)	34 (87%)	0
upper secondary	7 (37%)	12 (63%)	0
total	22 (26%)	63 (73%)	1 (1%)

Table 3. The relationships among source of programs using in math class

	both		only one		either
	(a)	(b)	(c)	(d)	(e)
1)- 2	28 (33%)	17 (20%)	35 (42%)	3 (4%)	65 (77%)
1)- 3	45 (54%)	22 (26%)	19 (23%)	12 (14%)	76 (90%)
1)- 4	42 (50%)	17 (20%)	22 (26%)	13 (15%)	77 (91%)
2)- 3	27 (32%)	19 (23%)	4 (5%)	34 (5%)	78 (92%)
2)- 4	35 (42%)	11 (13%)	6 (7%)	29 (35%)	58 (69%)
3)- 4	43 (51%)	28 (33%)	14 (17%)	13 (15%)	69 (82%)

(a) ( ) and 0

(b) usual-usual,s ometimes-sometimes, and once or twice-once or twice

(c) ( ) only

(d) 0 only

(e) ( ) or 0

[Appendix III]

Table 1. The relationships between teachers' conceptions

(a) yes-yes, no-no
(b) yes-yes, no-no, neutral-neutral
(c) ( )yes- 0no
(d) 0 no
(e) ( )no - 0 yes
(f) ( )no

	users						non-users					
	(a)	(b)	(c)	(d)	(e)	(f)	(a)	(b)	(c)	(d)	(e)	(f)
1)- 2	39	53	3	19	1	13	13	29	3	9	2	17
1)- 3	29	49	4	29	0	7	15	34	4	14	0	5
1)- 4	25	37	7	37	0	11	13	29	3	17	3	9
1)- 5	27	39	10	31	0	14	12	22	3	10	4	23
1)- 6	36	50	3	21	1	14	12	31	2	15	0	9
2)- 3	23	39	3	32	3	14	8	22	7	25	2	8
2)- 4	27	44	8	30	2	11	11	21	4	23	1	11
2)- 5	23	35	10	31	4	18	17	25	12	12	18	18
2)- 6	33	45	2	21	3	19	12	28	1	21	1	6
3)- 4	25	50	4	19	0	16	13	28	1	12	5	15
3)- 5	23	42	6	19	0	23	14	28	1	3	6	24
3)- 6	33	61	2	4	3	20	13	36	2	7	4	12
4)- 5	27	43	4	17	2	24	15	26	2	6	7	23
4)- 6	26	43	2	11	6	31	13	31	2	12	2	12
5)- 6	28	43	1	15	8	26	14	29	6	21	0	5

[user]

Table 2. Basic knowledge and skills

	elementary	lower sec.	upper sec.	total
yes	16	28	10	54
neutral	9	10	9	28
no	3	0	0	3
no answer	0	1	0	1

Table 3. Intuitions

	elementary	lower sec.	upper sec.	total
yes	12	26	12	50
neutral	11	12	5	28
no	5	0	2	7
no answer	0	1	0	1

Table 4. Logical thinking

	elementary	lower sec.	upper sec.	total
yes	12	16	5	33
neutral	10	21	13	44
no	6	1	1	8
no answer	0	1	0	1

Table 5. Possibility of problem solving

	elementary	lower sec.	upper sec.	total
yes	8	16	9	33
neutral	11	19	7	37
no	9	3	3	15
no answer	0	1	0	1

Table 6. Changes of problems

	elementary	lower sec.	upper sec.	total
yes	8	19	11	38
neutral	10	16	5	31
no	9	3	3	15
no answer	1	1	0	2

Table 7. Student's problem-solving

	elementary	lower sec.	upper sec.	total
yes	16	25	7	48
neutral	8	13	10	31
no	4	0	2	6
no answer	0	1	0	1

[non-user]

Table 8. Basic knowledge and skills

	elementary	lower sec.	upper sec.	total
yes	3	10	6	19
neutral	3	14	11	28
no	0	3	5	8
no answer	0	0	0	0

Table 9. Intuitions

	elementary	lower sec.	upper sec.	total
yes	2	12	10	24
neutral	4	10	11	25
no	0	5	1	6
no answer	0	0	0	0

Table 10. Logical thinking

	elementary	lower sec.	upper sec.	total
yes	3	7	2	12
neutral	3	12	12	27
no	0	8	8	16
no answer	0	0	0	0

Table 11. Possibility of problem solving

	elementary	lower sec.	upper sec.	total
yes	1	6	10	17
neutral	5	15	4	24
no	0	6	8	14
no answer	0	0	0	0

Table 12. Changes of problems

	elementary	lower sec.	upper sec.	total
yes	3	12	16	31
neutral	3	10	5	18
no	0	5	1	6
no answer	0	0	0	0

Table 13. Student's problem-solving

	elementary	lower sec.	upper sec.	total
yes	2	5	6	13
neutral	4	17	11	32
no	0	5	5	10
no answer	0	0	0	0

[Appendix IV]

Table 1. The relationships between source of programs and teachers' conception

(a) [yes(Q12)-often(Q7)] /[often(Q7)]

(b) [yes(Q12)-often+several(Q7)]/ [+several(Q7)]

(c) [yes(Q12)-often+several +once or twice(Q7)]/ [+once or twice(Q7)]

(d) [yes(Q12)-nothing(Q7)] /[not used(Q7)]

(e) [no(Q12)-often(Q7)]/[often(Q7)]

(f) [no(Q12)-often +several(Q7)]/ [+several(Q7)]

(g) [no(Q12)-often +several +once or twice(Q7)]/ [+once or twice(Q7)]

(h) [no(Q12)-not used(Q7)] /[not used(Q7)]

	YES (Q.12)				NO(Q.12)			
	user		non-user		user		non-user	
Q7-Q12	(a)	(b)	(c)	(d)	(e)	(f)	(g)	(h)
1)-1)	8/13	24/39	43/66	10/19	0/13	1/39	1/66	2/19
1)-2)	5/13	23/39	39/66	10/19	1/13	2/39	5/66	2/19
1)-3)	4/13	11/39	27/66	5/19	0/13	4/39	6/66	2/19
1)-4)	4/13	16/39	24/66	8/19	2/13	5/39	11/66	4/19
1)-5)	7/13	20/39	31/66	6/19	1/13	7/39	12/66	3/19
1)-6)	7/13	17/39	37/66	10/19	0/13	2/39	5/66	0/19
2)-1)	3/4	6/9	20/31	31/51	0/4	0/9	1/31	2/51
2)-2)	2/4	6/9	21/31	26/51	0/4	0/9	1/31	0/51
2)-3)	0/4	2/9	11/31	20/51	0/4	1/9	2/31	5/51
2)-4)	0/4	2/9	12/31	20/51	0/4	2/9	5/31	9/51

2)-5)	2/4	6/9	19/31	18/51	0/4	1/9	3/31	10/51
2)-6)	1/4	3/9	17/31	28/51	0/4	1/9	2/31	4/51
3)-1)	7/11	21/31	38/58	15/26	1/11	1/31	1/58	2/26
3)-2)	4/11	15/31	42/58	15/26	1/11	1/31	3/58	4/26
3)-3)	6/11	11/31	20/58	12/26	1/11	4/31	4/58	4/26
3)-4)	5/11	12/31	25/58	8/26	3/11	7/31	9/58	6/26
3)-5)	4/11	11/31	24/58	13/26	1/11	6/31	9/58	6/26
3)-6)	5/11	12/31	29/58	17/26	2/11	4/31	6/58	0/26
4)-1)	16/25	29/45	39/56	13/28	2/25	2/45	2/56	1/28
4)-2)	16/25	28/45	34/56	14/28	0/25	1/45	1/56	6/28
4)-3)	10/25	16/45	21/56	11/28	3/25	4/45	4/56	3/28
4)-4)	7/25	17/45	22/56	11/28	3/25	6/45	6/56	8/28
4)-5)	12/25	19/45	28/56	10/28	3/25	7/45	8/56	6/28
4)-6)	14/25	24/45	32/56	14/28	2/25	3/45	3/56	3/28

Table 2. The relationships between approaches to using computers and teachers' conceptions

	YES (Q.12)			NO(Q.12)		
	user		non-user	user		non-user
	(a)	(b)	(c)	(d)	(e)	(f)
1)-1)	18/25	47/73	6/11	2/25	2/73	1/11
1)-2)	14/25	42/73	7/11	1/25	5/73	2/11
1)-3)	13/25	27/73	5/11	2/25	7/73	1/11
1)-4)	15/25	29/73	4/11	3/25	12/73	2/11
1)-5)	14/25	33/73	5/11	4/25	12/73	2/11
1)-6)	13/25	41/73	6/11	3/25	5/73	1/11

2)-1)	3/8	39/58	14/26	0/8	1/58	2/26
2)-2)	4/8	36/58	13/26	0/8	3/58	4/26
2)-3)	2/8	23/58	9/26	0/8	3/58	5/26
2)-4)	5/8	25/58	8/26	1/8	8/58	6/26
2)-5)	7/8	29/58	9/26	0/8	9/58	5/26
2)-6)	4/8	34/58	13/26	0/8	2/58	4/26
4)-1)	1/4	13/21	40/63	2/4	2/21	1/63
4)-2)	1/4	11/21	38/63	0/4	1/21	6/63
4)-3)	1/4	9/21	23/63	1/4	2/21	6/63
4)-4)	0/4	6/21	27/63	2/4	5/21	9/63
4)-5)	1/4	7/21	31/63	1/4	6/21	8/63
4)-6)	1/4	12/21	35/63	1/4	5/21	4/63
5)-1)	2/5	23/34	30/50	2/5	2/34	1/50
5)-2)	2/5	20/34	29/50	0/5	1/34	6/50
5)-3)	1/5	10/34	22/50	2/5	4/34	4/50
5)-4)	1/5	13/34	21/50	2/5	5/34	8/50
5)-5)	2/5	15/34	23/50	2/5	5/34	9/50
5)-6)	2/5	17/34	30/50	2/5	2/34	4/50

---

user (a) [yes(Q12)-often(Q8)] /[often(Q8)]

(b) [yes(Q12)-often+sometimes(Q8)]/[often+sometimes(Q8)]

non-user (c) [yes(Q12)-never(Q8)] /[never(Q8)]

user (d) [no (Q12)-often(Q8)] /[often(Q8)]

(e) [no (Q12)-often+sometimes(Q8)]/[often+sometimes(Q8)]

non-user (f) [no (Q12)-never(Q8)] /[never(Q8)]

Table 3. Computer usage in the future

	active	passive	conditions	no answer
elementary	19 (67.9)	24 (7.1)	7 (25.0)	0 (0.0)
lower secondary	33 (84.6)	0 (0.0)	5 (12.8)	1 (2.6)
upper secondary	12 (63.2)	4 (21.1)	3 (15.8)	0 (0.0)
total	64 (74.4)	6 (7.0)	15 (17.4)	1 (1.2)

Discussion of Professors Sawada and Kumagai's paper:

Fey: Are there comments or questions to either speaker? Professor Fujii.

Fujii: I have a question about Table 11 on page 14; the third item attracts students. How was this asked in the questionnaire?

Kaida: Professor Fujii feels the number shown on the Japanese side is quite reasonable but the American one is extremely low and he doesn't know why. Can someone offer a reason for this low value?

Choate: I think it's mostly that American kids go to the school that's in their district and it isn't a factor of being attracted to a school because they have computers, and also most American schools have a lot of computers so, . . . Jim you want to argue with that?

J. Wilson: Well, there wouldn't very often be choices between a school that had lots and a school nearby that didn't have a lot of computers, so it isn't that decision.

Demana: In the process of collecting this data, looking at computer use in school, did you look at the use of hand-held calculators, or record the use of hand-held calculators?

Sawada: Well, in the case of the national survey, they surveyed only microcomputers, no hand-held calculators.

Demana: Do they have a conjecture about what the use is?

Sawada: Well, about 20 years ago, the use of hand-held calculators was encouraged, and, in fact, all the junior and senior high schools had the hand-held calculators. But since then, most of them are not being used, they're not welcomed. And it seems that in the case of the computers, it is quite different.

Kaida: I have a question. In the survey that was done with the total of 143 teachers, why that number?

Kumagai: Since the objective was to make a survey of the use of computers, they tried to find the schools which have computers and which frequently use computers; but since not very many schools have computers or use them, this is the number they had to work with.

Zilliox: I guess I would like to know if the researchers were shocked or surprised by any of the results that they got, or was this something that they anticipated? Or if there was any real awakening that they got from looking at the data?

Kumagai: We were more or less surprised at the response to question #12 which asks whether introduction of the computer would change in any way the types of mathematical problems; and the answer mostly was no, and the response was very low. My conjecture is that there were probably many math teachers that never used the computer to solve problems themselves. And if they did, probably the number of responses would change.

Choate: A question to Mr. Sawada. On page 7, the bottom paragraph speaks to the Japanese publication industry. One phenomenon in America that's occurring is the creation of software to go along with textbooks series and in our conferences that's a big selling point for the publishers. I have two questions about that for you. (1) is with the new curriculum, has the Ministry of Education stipulated that software has to go along with the textbooks?; and (2) in terms of the research, do you think that the responses would change if the teachers had experience with software that went along with existing text materials?

Sawada: With respect to the new curriculum or Course of Study just issued by the Ministry of Education two years ago, the actual practice will start in 1992 for elementary, 1993 for junior high, and 1994 for upper secondary schools. The textbooks going along with the new courses are being introduced only in the elementary schools now. None has been produced for lower secondary and upper secondary schools yet. Therefore, we cannot yet tell whether the software will go along with those textbooks or not. But certainly the Ministry of Education is recommending promotion of the software.

H. Wilson: This question is not about problem solving per se, it's more on teacher education. I guess for my own information, is there a program or an ongoing inservice requirement for teachers? Do they have to return to teacher education institutions for updating their knowledge. I'm concerned about changing their attitudes towards the use of the computers and whether it's going to be done with expertise or just put the teacher in the environment and expect the computers to be used in some way.

Sawada: Page 11, Table 9 shows the present condition of the teachers as far as computer knowledge is concerned. And recently in every prefecture, they give a seminar or training course especially for the lower secondary school industrial arts and homemaking teachers. Table 10 shows the location of the inservice training of teachers in computer education.

Miwa: This table shows only the percentage of the teachers attending the seminars. I would like to know the actual number of times inservice education is scheduled or offered.

Sawada: This table only shows the survey from 1989 onwards, and those who attended the inservice seminars and what percentage of the those attended were national or local or whatever. It doesn't say how many.

Fey: Maybe that's something you have to work out between yourselves. Mr. Morimoto, do you have a question?

Morimoto: Every summer this kind of inservice seminar has been held and so far what's going on is that the Ministry of Education invites teachers from every prefecture. They gather in a central place, get training and then return to their prefectures. And then they become instructors and give seminars to teachers in the local area. So far, it has started with the industrial arts and homemaking teachers, and is almost finished. Now they are including the science and math teachers. This year the inservice work in local areas is held at thirty different locations. But how many for math and science subjects, I don't know.

Becker: I'm curious about the implementation of computers in the new syllabus in 1994. Will there be difficulty in implementing the use of computers given the heavy emphasis on preparation for college and university entrance examinations?

Sawada: I hope something will change. Right now we have what we call the Association of Math Education, composed of college professors and math teachers. I'm a member of it. We meet twice a year and we're talking about including the computer as part of the entrance examination system.

Miwa: As you know, at the high school level and especially for general education, but not vocational, the curriculum and teaching methods is greatly influenced by the entrance examinations of colleges and universities. Now, this new Course of Study will start in 1994 which means that three years later the first graduates of high schools will be affected by this new problem. However, according to the new Course of Study, the computer is only optional, it's one of the electives and so it's a really small amount. So, if people say that the content from an elective should not be asked on entrance exams, then there will be the tendency for high school teachers to regard the computer very lightly. If that happens, it may even be possible that the computer is completely neglected at the high school level. So, at this moment, nobody can say anything about this. The strongest powers who have the greatest influence will be a group of college professors who determine the types of entrance examinations. So, we need to have good understanding, which we request, of the college professors regarding the use of computers.

Becker: Just out of curiosity, can students get instruction or experience in working with computers outside the regular schools (juku), or are there some other kinds of opportunities?

Sawada: In talking about the juku, there are three types: one is for advanced students who want to go to the good schools and to do that they have to take an entrance exam from elementary to lower secondary school, then from lower to upper secondary school, and then to college. The second level is for remedial purposes, for slow learners. Now, these are students who have either failed or have difficulty in the general public or general schools, and they need additional training. The third type is just for general students. Now, in either case though not a majority, some jukus use computers as an aid, but I don't know any school where the student can learn computer programming.

Morimoto: Well, it is true that the number of jukus using computers is very limited. However, in many cases the students have computers at home and are using the modem which connects the computer to a juku and then they study using computers. Or they can also purchase software from the juku and study at home.

Becker: How about in activity clubs in the schools, as a kind of extra-curricular activity?

Sawada: Well, according to the IEA research, it seems that computer usage in club activities is higher than for any other subject area.

Demana: Since the tests that determine whether the kids go to the university is decided by university professors, how do you change the university professors' way of thinking about this?

Sawada: It seems that we have almost no problems with the math education professors at the college level. The problem is with the math professors...not math education. So, if they can realize the importance of computers, then they will change their minds, and especially in very high level colleges and universities. If they change their system, then the other colleges will follow.

Fey: I noted that you begin all of that with the word "if." It seems now that it is time for lunch and an excursion. Jerry, do you want to describe the schedule for the afternoon?

Becker: Yes, now we'll go over for lunch and, afterwards, we'll visit the lab school. We'll be there approximately an hour to hear Gary Martin and his colleagues describe their geometry project. Then the bus will be right on time for the excursion...Hawaiian time, that is.

End of Discussion

COMPUTER USE IN TEACHING MATHEMATICAL PROBLEM SOLVING: PRE-SERVICE TEACHER EDUCATION IN YOKOHAMA NATIONAL UNIVERSITY

Yoshihiko Hashimoto
Faculty of Education, Yokohama National University
Yokohama, Japan

1. Introduction

My title given by the Japanese side is pre-service teacher education in our University related to computer use in teaching mathematical problem solving.

Education School	elementary	secondary
preservice	A	B
in-service	C	D

The above figure is useful for analysis. We can think about 4 cells such as A (pre-service) education at elementary school level), B (pre-service, secondary), C (in-service, elementary) and D (in-service secondary). I would like to provide information about B in this paper. First of all, let's think about the following problem. (Johnson, 1985; Borasi, 1986) A student is given the fraction 16/64 to reduce.

We can see the same digit in both the numerator and denominator, so we can cancel like this:

$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$$

This is a case in which an incorrect mathematical procedure led to correct results. The problem is that "Are there any two digit numbers in both numerator and denominator to satisfy the condition?" We can find nine trivial answers easily. Others can't be found intuitively.

We can think of computer use to solve this problem. The programming and output written here were made by one of my undergraduate students. He could get the correct answers by using the computer, after learning the basic language in only several hours, as follows.

```
 10  PRINT  "Reduction in special case"
 20  PRINT TAB(1) "A"; "   ";  "B"; "   ";  "C"; "   ";  "AB/BC"
 30  FOR A=1 TO 9
 40  FOR B=1 TO 9
 50  FOR C=1 TO 9
 60  IF (10*A+B)*C=A*(10*B+C) THEN PRINT A;B;C; "   ";10*A+B"/ "10*B+C
 70  NEXT C
 80  NEXT B
 90  NEXT A
100  END
```

```
Reduction in special case
 A B C   AB / BC
 1 1 1   11 / 11
 1 6 4   16 / 64
 1 9 5   19 / 95
 2 2 2   22 / 22
 2 6 5   26 / 65
 3 3 3   33 / 33
 4 4 4   44 / 44
 4 9 8   49 / 98
 5 5 5   55 / 55
 6 6 6   66 / 66
 7 7 7   77 / 77
 8 8 8   88 / 88
 9 9 9   99 / 99
```

If one thinks about the problem using paper and pencil, then it will take much more time. If students can translate this problem into "BASIC" language --- in this situation, FOR ~NEXT, IF~ THEN, PRINT ---, then they will be able to get 13 correct answers, including (numerator, denominator) = (16, 64), (19, 95), (26, 65), (49, 98). I would like to emphasize that it is effective to use repetition or iteration as one of the characteristics in the function of computer.

I think it is a good example of computer use in mathematical problem solving. The important thing is that we have to give "good" problems in mathematical problem solving to learners. Such problems should be developed and prepared.

2. Pre-service Teacher Education in Our Faculty

Pre-service teacher education at the secondary school level in the faculty of education at Yokohama National University will be explained. We have four faculties such as engineering, economics, management, and, education.

I have three classes in the undergraduate course. One of them is a required subject related to teaching methods (90min. x30). I treat the nature of the classroom process, various types of teaching including their merits and demerits, making lesson plans and use of technological equipment and so on in the course.

I treat at least two things about computers in the course. They are reduction in special case as I explained now and tessellation in plane figures related to the use of technological equipment. Tessellation will be shown practically in the Seminar. I am convinced that using the computer is very effective and efficient for movement of figures. The software was made by one of my graduate students in the master course (Appendix 1). The student took about two months or so to make the software by using Quick BASIC. Of course, I gave him some advice from a mathematical and an educational point of view.

3. School Teacher's Certificate

The requirements for a School Teacher's Certificate were revised in March, 1989 by the Ministry of Education. At least 20 credits (2 credits means 90min. x15) for certificate of lower secondary school(grades 7-9) in mathematics must include the following.

Algebra	6 or 4
Geometry	6 or 4
Analysis	4
Probability, Statistics	4 or 2
Computer	2

At least 20 credits for certificate of upper secondary school(grades 10-12) in mathematics must include the following.

Algebra	6 or 4
Geometry	6 or 4
Analysis	6 or 4
Probability, Statistics	4 or 2
Computer	4 or 2

One of the characteristics in the revised school teacher's certificate for becoming a mathematics teacher was that the computer instead of survey was introduced. The new school teacher's certificate is applied from sophomore students in 1991. Taking four credits for computer was determined by faculty members in the department of mathematics education and the department of mathematics in our University. That is implemented from last April. One of my colleagues teaches computer.

4. "Introduction to Computer" in our Faculty

This subject is a required subject for sophomore students majoring in mathematics in our Faculty. Four credits are given. This means 90 minutes/week 30 times. The number of students is about 50. Lectures and practice are carried out about half-and-half by using about 30 laptop personal computers.

The main purpose of this course is to teach fundamental and important algorithms in computer programming by using "PASCAL". Of course, it is important for students to be able to understand and use it. The outline is as follows:

a. Explanation of hardware in computer
b. Numerical expression and errors in computer
c. Explanation of PASCAL's grammar and important algorithms used in basic sentences
d. Practice of programming by mathematical problems

The mathematics curriculum for the upper secondary school (grades 10-12) was revised by the Ministry of Education in 1989 and will put in force beginning in April, 1994 (Appendix 2). In the curriculum, the content of computer is included in the subjects "Mathematics A", "Mathematics B" and "Mathematics C". Therefore, the following things are treated in the "Introduction to Computer":

a. Greatest common divisor by the Euclidean algorithm
b. Calculation of square root by iteration
c. Solution of non linear equations by Newton's method or method of bisection
d. Numerical integration by the trapezoidal rule or Simpson's rule
e. Binomial coefficient by using Pascal's triangle.

(This section was prepared by associate professor Yutaka Baba who is one of my colleagues and a statistician.)

5. Future Perspective

As I mentioned earlier, the school teacher's certificate and the mathematics curriculum were revised. According to these, we have been implementing a subject "Introduction to Computer" since April 1991 in our Faculty as a required subject for mathematics students. I hear that some other universities assign two credits for junior year students. My feeling is that if a consensus on computer use between classroom teachers and board of education emerges, the Japanese will move quickly to implement more widespread use of computers. The time has come for us to introduce computers in school mathematics in Japan.

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$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$$

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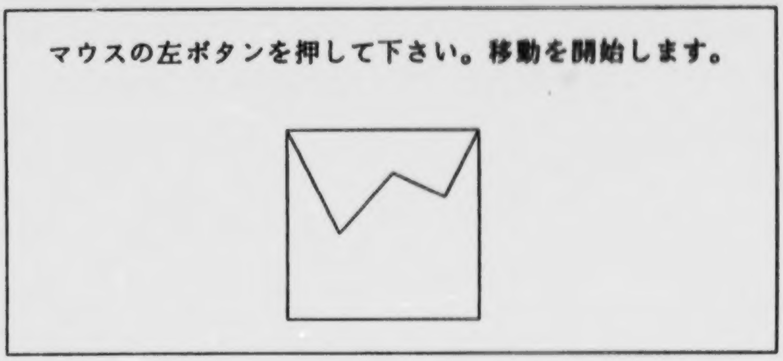
Algebraic and Numerical Explorations Inspired By The Simplification :

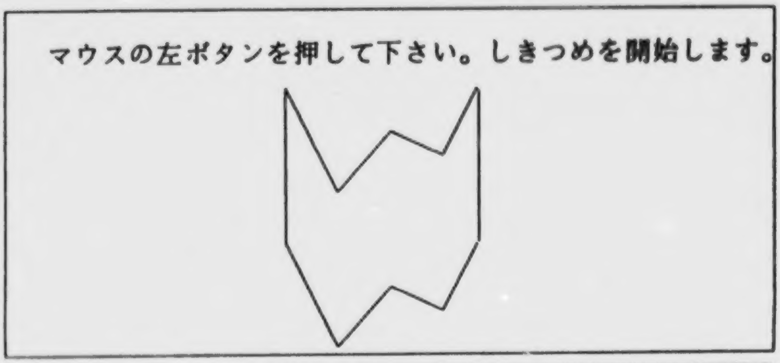
$$\frac{1\cancel{6}}{\cancel{6}4} = \frac{1}{4}$$

Focus on Learning Problems in Mathematics, Vol. 7 Numbers 3 and 4, 1985, pp. 15-28.

Appendix 1. Tessellation


マウスの左ボタンを押して下さい。移動を開始します。

Push the left button of the "mouse." Tessellation will start.


マウスの左ボタンを押して下さい。しきつめを開始します。

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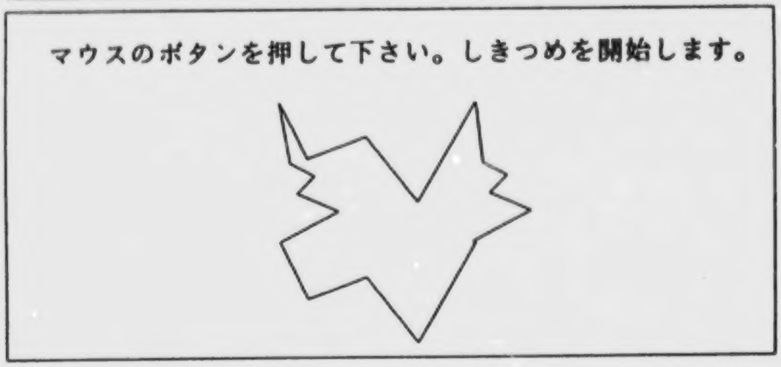
マウスのボタンを押して下さい。しきつめを開始します。

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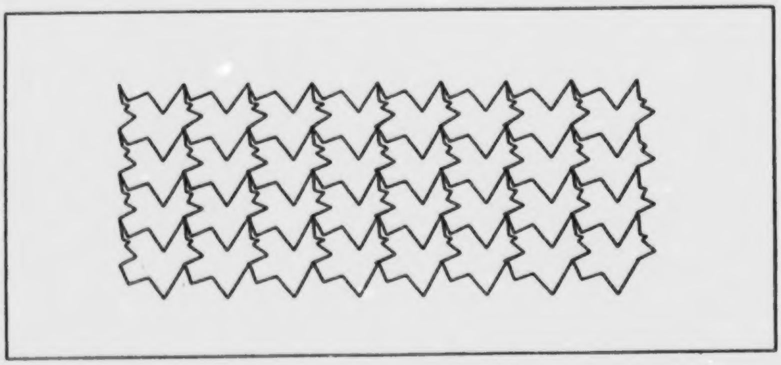

Appendix 2. Mathematics Curriculum in Upper Secondary School

Mathematics in the upper secondary school is composed of several subjects whose titles and associated credits are as shown below.

Mathematics I (Standard number of credits 4)
Mathematics II (3), Mathematics III (3)
Mathematics A (2), Mathematics B (2)
Mathematics C (2)

Mathematics I is required for all students, but the other mathematics subjects are optional. One credit consists of 35 class hours and a class period of 50 minutes is defined as one class hour.

Mathematics I
1. Quadratic Functions 2. Geometrical Figures and Mensuration
3. Treatment of Number of Cases 4. Probability

Mathematics II
1. Various Functions 2. Geometrical Figures and Equations
3. Variation of Values of Functions

Mathematics III
1. Functions and Limits 2. Differential Calculus 3. Integral Calculus

Mathematics A
1. Numbers and Algebraic Expressions 2. Plane Geometry 3. Sequences
4. Computation and Computer

Mathematics B
1. Vectors 2. Complex Numbers and Complex Number Plane
3. Probability Distribution 4. Algorithm and Computer

Mathematics C
1. Matrix and Linear Computation 2. Various Curves
3. Numerical Computation 4. Statistics

The underline parts are related to computer use.
(Ministry of Education, translated by Nagasaki, Sawada and Senuma(National Institute for Educational Research); Mathematics Program in Japan, 1989)

Discussion of Professor Hashimoto's paper:

Choate: Now let's open the floor for discussion. Professor Becker.

Becker: First, I would like to compliment Professor Hashimoto for a very concise and clear presentation. Professor Hashimoto, you listed five items towards the end of your talk. Could you repeat the last two of those? I didn't get all of them down.

Hashimoto: I'm sorry. The last two things are numerical integration by the trapezoidal rule or Simpson's rule, and the last is the binomial coefficient using Pascal's triangle.

Becker: Okay, then, maybe one more question. What were these examples of, were they examples of ideas being taught using the computer?

Hashimoto: They are included mainly in elective course C. Of course, they are also included in both mathematics A and B, but mainly in C where they are emphasized.

Damarin: Sticking with those same items, do you have your students program solutions to those, or do you have demonstration programs, of, for example, the greatest common divisor by the Euclidean algorithm? How do your teacher education students use the computer in relation to that problem?

Hashimoto: Please look at page 40, and 41 also. This course is taught by my colleague so I don't know in detail what is going on. However, as far as the Euclidean algorithm is concerned, as stated on page 41, #3, writing the program is expected.

Fey: I think people in the U.S. who teach methodology courses for preservice teachers find it a frustrating experience because it's hard to convey ideas of how to conduct a classroom out in the school when you're not in the school showing it. How do you run your lessons about computer use? Do you try to simulate a classroom environment and teach the preservice teachers the way you would like them to teach their students? Or do you give a lecture about how to use this tessellation program? How do you create the kind of classroom atmosphere that you want with computers?

Hashimoto: In my class, luckily having only about 15 students, I am recommending group work and give one terminal to every two students. And then, providing software, they can experiment themselves and advance as they like at their own pace.

Demana: Let's assume for a minute that you do get computers integrated into your classrooms and you're doing it widespread in your mathematics education courses. What impact do you think that will have on the mathematicians at the university and do you have any past history of where the material on the entrance requirements and what happens in the mathematics department has been impacted on by what you've done in school mathematics and mathematics education?

Hashimoto: That question was answered by Professor Miwa yesterday, so maybe this is repetition again. But by 1994, the new course will be set up and by then it depends mainly on how the prestigious or prominent universities will deal with the type of entrance exams. Especially in the math teachers' training, the importance is the relationship between the mathematicians and math educators among the college professors. In my case, at my university, there is a very good relationship and the math professors do understand what the math educators want or are aiming at.

Miwa: I just want to add to my comment yesterday, that the curriculum is set up nationally. It is revised about every ten years, which is a very slow pace compared with the very rapid change in the world. So, for example, in the case of computers, it's just at this time that the computer is being introduced but not in a very progressive way. I think if computers were not introduced in the revised Course of Study, probably they wouldn't be introduced into school education on a full scale until the 21st century.

Dugdale: Concerning the tessellations program, you mentioned that making the tessellation was only the first part of the activity and that you expected a proof of why that figure would tessellate. Can you say more about this? For example, how much formality do you expect in a proof, how much variety do you see, and how do students approach the task?

Hashimoto: As far as the introduction of tessellations is concerned, our plan is to spend three class periods of 90 minutes each. In the first period students work on the regular polygon and determine which regular polygons tessellate. Certainly there are only three, the triangle, square, and hexagon. Then in the next period, we treat more than two regular polygons and there are eight possible combinations. And then we go to any type of polygon, in varying possibilities, and certainly any quadrilateral will tessellate. However, for the students, this is very difficult to understand - why any type of quadrilateral will tessellate. Of course, in that case students use manipulatives and cut out papers and try to work it out and see why. Then they go to the general cases using the computer. Since this software includes the five ways of tessellating and using the idea of congruent, it is rather easy to understand; and that's how it goes. This is included especially in the math A curriculum (as stated on page 38), and math A includes plane geometry in which students learn transformations using the computer. We have developed this course in relation to this curriculum.

Miwa: This is related to Professor Fey's narrative question given before. When you talk about the computer in teaching, there are two things involved. One is to teach about the computer itself and another one is to teach something using the computer. When you teach computer science, of course you have to use the computer. Now I know you have a lot of experience in teaching in preservice training. Could you tell us the how you are doing in your situation? How to introduce computer education in preservice training?

Fey: Well, one of the things we try to do is to simulate lessons with the prospective teachers and to teach topics from the secondary school curriculum using the technology the way we would use it. It's not always effective because preservice teachers are very skeptical. They think they know how to teach and so it's hard to create the atmosphere that you want. Other times we'll, perhaps not very effectively, give them a piece of software like the Geometric Supposer or Sketchpad, show them some things about what it'll do, and then ask them to go to the computer and explore the tool and come up with some ideas of their own, perhaps then the task will be to design lessons that would use this technology themselves. So, some of it is simulated lessons, some of it is introduction to a tool, and then a lab assignment where the idea is to figure out how they would use it in teaching after we've given some examples.

Zilliox: I guess I'd like to add something to what Professor Fey said. It may just, I don't think it just applies to the program that we have at the University of Hawaii, but one of the expectations we have of preservice teachers is that they become personal users of the computer so that their writing assignments have to be done in word processing; so, that they overcome some of the fears that they have of just the equipment. And the second thing is that we expect them to learn mathematics themselves through using the computer. It's not just how the students they may teach would learn mathematics, but their learning mathematics themselves through various pieces of software.

Sawada: Professor Hashimoto talked about computer use in tessellation. Related to this, I would like to ask how often the computer is used in math education at the college level. One of the reasons the computer is not often used in the lower or upper secondary school levels is because the computer is not very often used at the college level or at the teacher education level.

Hashimoto: In my university, we use, in 25 classes, 2 to 3 class periods for work on computers. In other words, about ten percent. However, starting in the sophomore year and in some schools including the junior year, this will change the situation a lot more. But, at present, in general I can say that certainly the computer is very rarely used at the college level. Of course, in a college of liberal arts an elective on computers can be chosen; but not very many students choose it.

H. Wilson: I have a question on the issue of certification. Is the certification standard established for national norms, or is it very much a function of the university? Is it on a common basis that all this is changing or is it just at your university?

Hashimoto: It's on a national level.

Becker: I would like to go back and add to what Jim Fey was saying about how we make use of computers in preservice education, and I'll talk just about the example of Southern Illinois University at Carbondale and its course called "Methods of teaching elementary school mathematics." First of all, a little background on a typical student in the preservice training course. Students' mathematical background almost always is not very substantial. Perhaps they have had junior high school mathematics, some have had algebra I also, and some may have had a geometry course in secondary school. Not frequently have they had a full sequence of math courses through upper secondary school. Then they come to our university where they are required to take two mathematics courses: one is an informal development of the real number system, and the other is informal geometry although sometimes that course is taught by professors who teach a kind of a tenth year secondary school deductive geometry course. Generally, we don't like that but some teach it that way nevertheless. And most of the students have had no contact with computers. Now, in our college of education we have two microcomputer labs that are equipped with Apple IIe microcomputers and now we are converting to Macintoshes. In the elementary mathematics methods course there are two components of what we do with computers. First, we deal with LOGO a little bit, in about five to ten 50-minute periods. We begin by performing physically some of the movements of the turtle in the classroom. And in doing that we introduce maybe 8 to 10 simple commands in LOGO. Then we go to the microcomputer lab and students use sample lessons that come from the Minnesota Educational Computer Consortium. The students actually work through some of those lessons. In that way we try to provide a bridge from what we are doing at the university level to what they will be doing in the school classroom. We also solve some problems by writing a program in LOGO. We discuss how they might be able to integrate what they have learned about LOGO into their own teaching. The second component involves software. We have some software from Sunburst Communications. We have the students sit at the microcomputer and work through some of the software. Later we discuss how they might use this software in their classrooms. And, finally, I should mention that it's likely that when these teachers are out teaching in the classroom they will not have easy access to computers. They might be able to get one or two computers that the school has for use in their classrooms some of the time.

Teague: Thank you very much. It is now break time. I'd like to thank Professor Hashimoto for a very interesting and stimulating presentation.

End of Discussion

GRAPHICAL REASONING FOR NEW APPROACHES TO TOPICS IN MATHEMATICS

Sharon Dugdale

University of California at Davis

Introduction

Use of function plotting software can facilitate mathematical reasoning and visualization of functional relationships. However, in order to take full advantage of this capability, new instructional approaches need to be devised. The examples discussed in this paper were developed to explore the potential of function-plotting tools to promote graphical reasoning in two contexts. The first context, trigonometric identities, is a topic that traditionally does not involve graphical representations. Here the goal was to integrate graphical representations into the topic and to compare the results of two different approaches--one approach that required graphical reasoning, and one that did not. In the second context, polynomial functions, the goal was to replace a traditional rule-based treatment of polynomial graphs with a computer-interactive treatment, using graphical reasoning to develop a sense of why polynomial graphs behave as they do.

Building a Foundation for Trigonometric Identities

Trigonometric identities are traditionally dealt with as exercises in symbol manipulation. However, guided experimentation with trigonometric graphs can help students acquire fundamental understandings and facility with the relationships among these functions in a possibly more transparent and intuitive context. Two instructional approaches were developed to incorporate graphical representations into a unit on trigonometric identities. One approach used graphical representations as the *foundation* for trigonometric identities. The other approach *supplemented* a traditional textbook treatment of trigonometric identities with related graphing activities. A study was conducted to compare students' performance using the two approaches. A more complete description of the instructional approaches and the comparison study appears in Dugdale (1990).

The Graphical Foundation Approach

In the Graphical Foundation Approach (Figure 1), students began with guided graphical exploration, observing relationships among functions. When graphical exploration suggested particular relationships among the trigonometric functions, students verified the observed relationships by algebraic symbol manipulation. Formalization of proving trigonometric identities followed the informal observations and verifications of equivalences among functions.

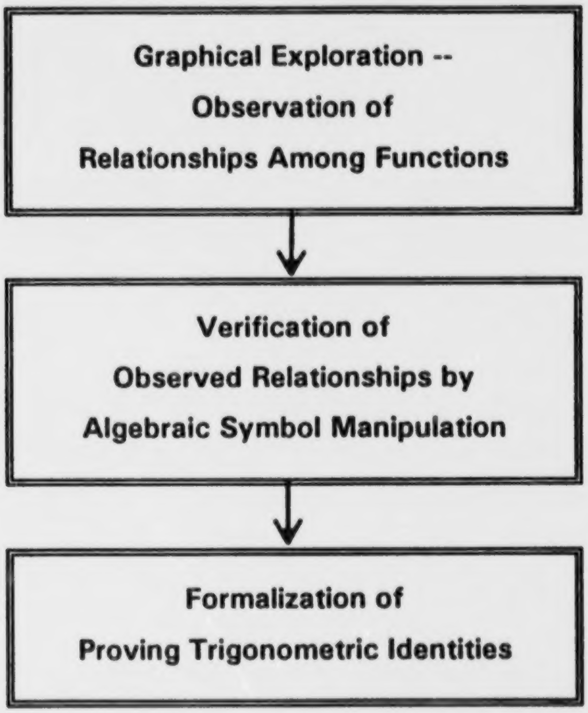
Graphical Exploration -- Observation of Relationships Among Functions
Verification of Observed Relationships by Algebraic Symbol Manipulation
Formalization of Proving Trigonometric Identities

Figure 1. The Graphical Foundation Approach.

The instructional unit included a variety of activities, many of them requiring students to use the graphs of functions to predict graphically the shapes of other functions before plotting. For example, students used the microcomputer to plot the graphs of $y = \cos(x)$ and $y = \csc(x)$, then predicted what the graph of the product function $y = \cos(x)\csc(x)$ would look like. Students were provided transparent slides (like those used with overhead projectors) to cover the microcomputer screens. Using overhead projector pens to draw on the transparent slides, students added their own work directly to the microcomputer display. Figure 2 shows a typical markup and prediction.

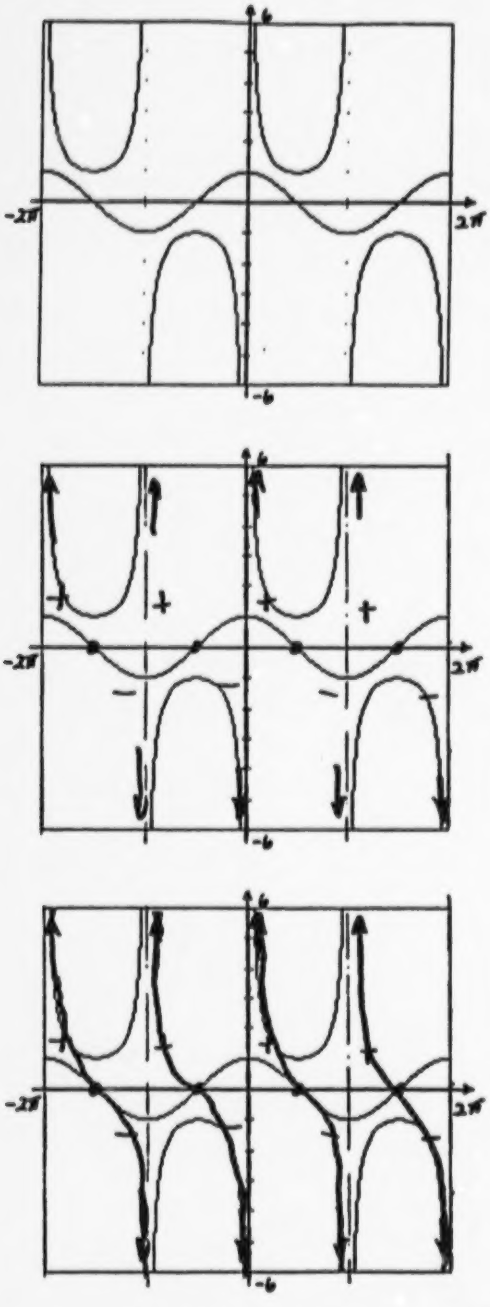

Figure 2. In the first frame, a student has plotted $y = \cos(x)$ and $y = \csc(x)$ and has positioned a transparent slide over the microcomputer display, matching the axes on the slide with the axes on the screen. In the second frame, the student has used the graphs of $y = \cos(x)$ and $y = \csc(x)$ to determine key features of the graph of $y = \cos(x)\csc(x)$. In the third frame, the student has sketched a prediction for $y = \cos(x)\csc(x)$.

In predicting the graph of $y = \cos(x)\csc(x)$, students typically noted the zeros of $y = \cos(x)$ and marked those places as zeros also of $y = \cos(x)\csc(x)$, reasoning that the product is zero wherever one of the factors is zero. A likely next step was to note the asymptotes of $y = \csc(x)$ and mark those as x-values for which $\cos(x)\csc(x)$ is also undefined. (These x-values were usually marked with vertical lines in a different color, to indicate that there were no function values on those lines.)

Students also noted the regions of positive and negative values for $y = \cos(x)\csc(x)$. Within each interval bounded by a zero and a discontinuity, if two functions are either both positive or both negative, their product is positive. If one function is positive and the other negative, their product is negative.

By examining the two given graphs near the marked discontinuities, students could predict the asymptotic behavior of $y = \cos(x)\csc(x)$. At each of the x-values for which $\cos(x)\csc(x)$ is undefined, $\cos(x)$ is near either 1 or -1, and $\csc(x)$ is either very "large positive" or very "large negative." Hence, the product is either very "large positive" or very "large negative." After making various observations of this type and marking their displays appropriately, students sketched a prediction for $y = \cos(x)\csc(x)$. Students checked their predictions by having the microcomputer plot the predicted graph.

Most of the graphs predicted and then checked by plotting were chosen to be easily recognizable, once they had been plotted. For example, the graph predicted in Figure 2 is equivalent to $y = \cot(x)$. After identifying the graph as $y = \cot(x)$, and plotting to verify, students were asked to justify algebraically the observed equivalence between the predicted function and the function it turned out to look like. Students needed to decide what of their previous knowledge was applicable and to devise a convincing argument for the apparent equivalence. Beginning with simple identities, such as $\cos(x)\csc(x) = \cot(x)$, as used in the example, students could readily recognize that substituting the definitions of functions and manipulating the resulting expressions would verify that the two functions were equivalent.

After verifying an equivalence, students were sometimes asked to produce another function which would make the same graph as the two functions that had been proved equivalent (for example, another function that would make the same graph as $y = \cot(x)$ and $y = \cos(x)\csc(x)$, in the example above). By using further substitutions and manipulations, students found various ways to make the same graph.

In using two graphs to predict the shape of a related function, students used and shared a variety of ideas, for example:

- When one graph has a function value of 1, the product is the corresponding point on the other graph. (And similarly, when one graph has a function value of -1, the product is the *opposite* of the corresponding point on the other graph.)

- When, within an interval, one graph has function values *very near* 1, the product within that interval is *approximated* by the other graph.
- Where two graphs cross (that is, their function values are equal), the quotient is 1.

Useful observations were easy to make, because the basic trigonometric functions are periodic, with frequent zeros, asymptotes, and relative maxima and minima of values 1 and –1.

Some activities involved the identity $\sin^2(x) + \cos^2(x) = 1$, which had been previously encountered in the context of sine and cosine as the coordinates of a point on a unit circle. For example, students were asked to plot the graphs of $y = \sec^2(x)$ and $y = \tan^2(x)$, as shown in Figure 3. Then by comparing the two graphs, students were requested to write an equation to express the relationship between the two functions and then to prove algebraically that their equation is always true. With some discussion among pairs or groups of students, the observed graphical relationship was described by the equation $\sec^2(x) = \tan^2(x) + 1$ (or some variation). The relationship was then verified by substitution and symbol manipulation involving the identity $\sin^2(x) + \cos^2(x) = 1$.

Following this basis of experience, proof of trigonometric identities was formalized and practiced in the context of the usual textbook exercises.

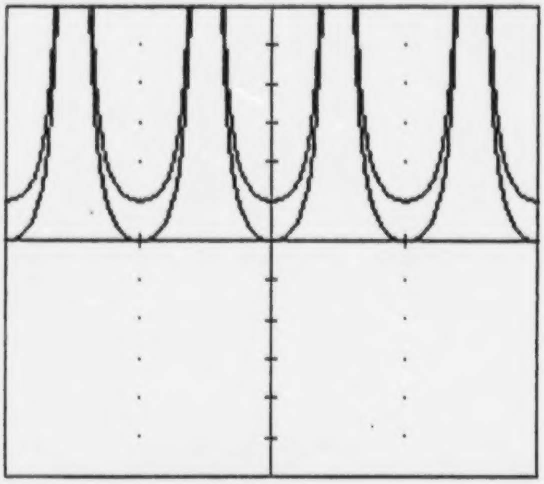

Figure 3. Students were asked to plot the graphs $y = \sec^2(x)$ and $y = \tan^2(x)$, then write an equation to express the relationship between the two functions.

The Supplemented Traditional Approach

The Supplemented Traditional Approach (Figure 4) followed the usual textbook approach to trigonometric identities, with a focus on following examples and practicing procedures. The teacher introduced the eight Fundamental Identities given in the textbook (Dolciani, Wooton, Beckenbach, & Sharron, 1980, pp. 545-546), showed examples of how to use the Fundamental Identities to prove identities in the textbook exercises, and assigned exercises for the students to practice. In conjunction with these exercises, students participated in the following graphing activities related to identities:

- Plotting a variety of trigonometric functions and recording equivalences.
- Simplifying trigonometric expressions and using graphs to verify the equivalence of the original expression and the simplified expression. This technique was useful for finding errors in symbol manipulation--if the graphs showed the original and final expressions to be not equivalent, intermediate steps could be graphed in order to locate errors.
- Graphically determining whether given equations were identities. Students were asked to prove those that graphically appeared to be identities.

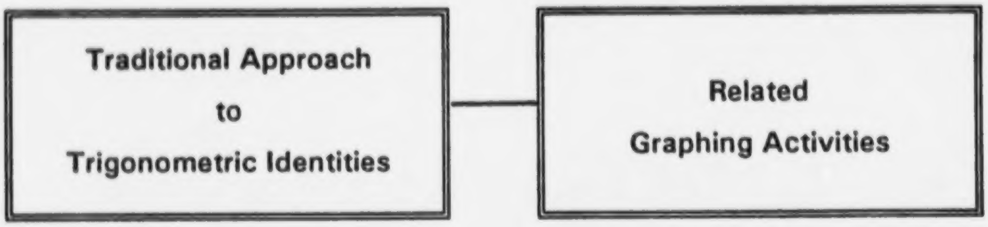

Figure 4. The Supplemented Traditional Approach.

The Comparison Study

A study was imbedded in the normal classroom instructional sequence for trigonometry. The subjects, 30 students in grades ten through twelve, had completed the introductory trigonometry material from their textbook (Dolciani, et al., 1980, Chapter 14) and were ready to begin identities. Subjects were divided randomly into two treatment groups. One group used the Graphical Foundation (GF) approach; the other, the Supplemented Traditional (ST) approach.

Results of a posttest indicated no significant difference in the two groups' performance on the standard content of proving trigonometric identities, but a significantly higher ($p = .010$) posttest performance for the Graphical Foundation Group on relating trigonometric functions to their graphical representations. Complete statistical results and further discussion are presented in Dugdale (1989).

As might be anticipated, the initial symbol manipulation work of ST subjects was generally more standard than that of GF subjects. GF subjects exhibited more variety and personal involvement in their methods. There was also a noticeable difference in the graphs drawn by the two groups during their computer activities. ST subjects tended to produce more uniformly neat, accurate, and properly labelled graphs. In contrast, GF subjects were more likely to produce sketchy, sometimes incomplete, graphs. ST subjects appeared to regard their graph drawing as the final goal of the activity, whereas GF subjects approached the task more as scratch work on the way to a solution.

GF subjects showed higher posttest performance in relating functions to their graphical representation, despite the fact that ST subjects were exposed to *more* graphical representations. For example, ST subjects drew the graphs of 18 different equations, 6 of which were on the posttest. In contrast, GF subjects drew the graphs of only 7 equations, one of which was on the posttest. ST subjects did *routine* work with *many* graphs, while GF subjects were involved in more *thoughtful* work with *fewer* graphs.

In addition to using graphical representations as the foundation for trigonometric identities, the Graphical Foundation Treatment was intended to involve students in:

- Experiencing active participation in the development of mathematical ideas. Students were to predict and figure out, rather than follow examples, copy graphs, and have ideas explained.
- Building a qualitative perspective before formalizing procedures. Trigonometric identities were introduced graphically, and the usual symbol manipulations were used to verify the relationships observed in the graphing activities.
- Applying previous knowledge and skills to a current problem without being told what, in particular, to do. Students were to decide what of their previous knowledge was applicable and devise convincing arguments for observed equivalences. Students were involved in learning more generally-applicable inquiry techniques in addition to basic content.

The results of this study suggest that a graphical reasoning approach, with careful attention to students' experiences beyond the immediate content goals, can produce a richer learning experience without significant detrimental effect on the mastery of standard content.

Visualizing Polynomial Functions

Although polynomial functions are often expressed graphically, and the graphs are sometimes used effectively in solving problems, there is little mathematical reasoning apparent in standard treatments of the relationship between the terms of a polynomial function and its graph. In courses prior to Calculus, graphs of polynomial functions of degree greater than 2 are typically approached with a set of rules, most of which must be accepted on faith. (See, for example, Brown & Robbins (1984, pp. 62-64) and Wooton, Beckenbach & Fleming (1981, pp. 213-219).)

A computer-interactive approach to polynomial graphs was developed to reduce emphasis on memorized rules in favor of qualitative understanding of functional behavior, visualization of functional relationships, and graphical investigation of mathematical ideas. The Monomial Sums Approach is based on the following ideas:

- Just as a polynomial is a sum of terms of the form ax^n, the graph of a polynomial function can be visualized as the sum of the graphs of monomial functions $y = ax^n$.
- The effect of each term of the polynomial function can be predicted in the graph of the entire function.
- The lowest-exponent term dominates the graph nearest the y-intercept (where $x = 0$), each higher-exponent term shows its effect in turn as x increases in absolute value, and the highest-exponent term dominates the extremes.

Some of the usual rules for polynomial graphs, such as dominance of the leading term, are readily apparent within this approach. Ideas about turning points, x-intercepts, and symmetries, as well as some of the more advanced theorems (for example, Descartes' Rule of Signs) are accessible through investigation. A more extensive discussion of the Monomial Sums Approach and its use appears in Dugdale, Wagner, and Kibbey (in press).

Approaching polynomial graphs as sums of monomial graphs is an old idea. (See, for example, Gibson (1905, pp. 108-110).) The use of such a method today is not motivated by the need to construct the graph of a function. Rather, in conjunction with appropriate function plotting software, visualizing polynomial graphs as sums of monomial graphs can be a convenient conceptual aid for understanding the behavior of these functions and their graphs.

The Monomial Sums Approach and associated software evolved over the course of two school years, with the participation of high school students enrolled in Advanced Algebra and Analytic Geometry classes. The students' classwork had already included some introduction to graphs of polynomial functions. Before beginning the Monomial Sums Approach, students were observed to depend largely on memorized rules for relating polynomial functions to their graphical representations. When the rules were not clearly remembered, students would ask someone the rules or look up the rules. In cases where the available rules offered no help, students resorted to random guessing.

To begin visualizing polynomial functions as sums of monomial functions, students were asked to use the graphs of two monomial functions to predict the graph of their sum. For example, after using the microcomputer to plot the graphs of two monomial functions, $y = x^5$ and $y = x^2$, students could be asked to predict the graph of their sum, $y = x^5 + x^2$. Comparing their predictions to a computer-plotted graph of $y = x^5 + x^2$, students could see which features of the graph they had predicted well and which features they may have missed.*

The purpose of the prediction activity was not to build proficiency with adding coordinates, but, rather:

- to lay a foundation for thinking of polynomial functions as sums of monomial functions, and
- to observe that the resulting function is dominated by the lower exponent term near the origin, and by the higher-exponent term for larger absolute values of x. (See Figure 5.)

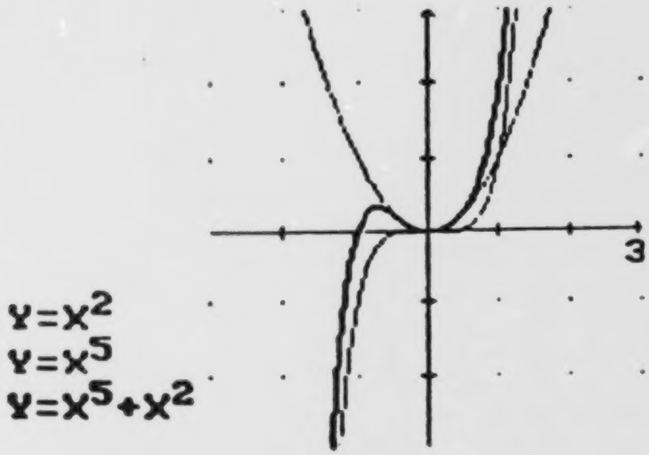

Figure 5. The graphs of two monomial functions, $y = x^2$ and $y = x^5$, and the graph of their sum, $y = x^5 + x^2$. The sum is dominated by the lower exponent term near the origin, and by the higher-exponent term for larger absolute values of x.

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*Note that this is not the best context in which to *introduce* the notion of using two graphs to predict the shape of a third graph, because polynomial graphs quickly become very steep as x increases in absolute value. Students who had experienced prediction with other types of graphs (for example, the trigonometric graphics discussed earlier) were better prepared to deal with this activity.

The dominance of the *lower*-exponent term was surprising to students, but the reason for it was apparent from the prediction activity--for x-values near zero, the higher-exponent term is nearly flat, so that its contribution to the sum is negligible. The implications of higher exponents producing smaller function values for x between -1 and 1 was more persistently troublesome for students than the other ideas involved in building sums of monomial functions. For example, it could be perplexing to notice evidence that x^2 is sometimes greater than x^4, even after discussing and using this property in earlier activities. By using this idea throughout their work with polynomial graphs, students became accustomed to its reality, although encounters in various contexts seemed necessary in order for the concept to "fill out." In general, progress with concepts seemed to depend on thinking about the graphs from various perspectives, questioning seeming contradictions, relating current instances to previous observations, and working toward an integrated understanding.

Through prediction activities and guided exploration, students investigated the effects of individual terms of a polynomial on the overall graph of the polynomial function. Students also constructed polynomial functions to produce desired features in graphs. For example, in Figure 6 a graph is shown, and students are asked to build a polynomial function to match the graph. For this type of activity, the Monomial Sums Approach facilitates a solution process beginning with the lowest-exponent term and working outward from the y-axis, leaving the highest-exponent term until last, as shown in Figure 7.

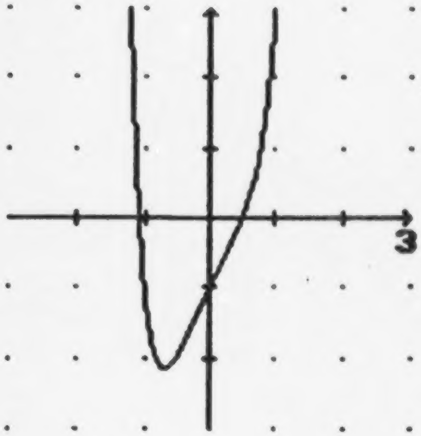
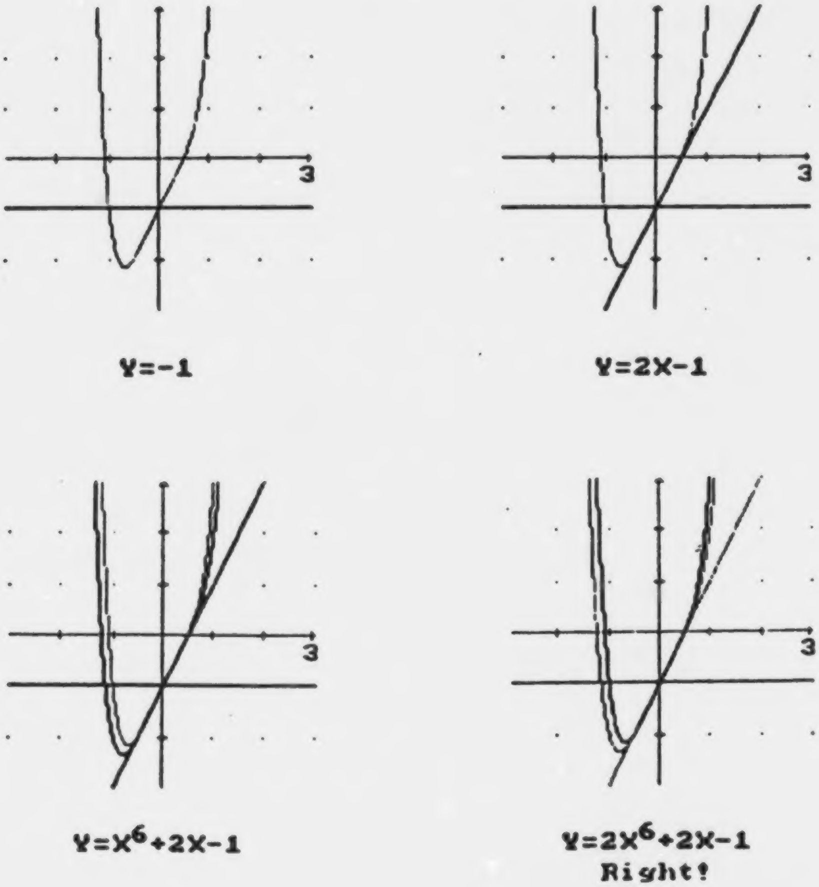


Figure 6. Students are asked to build a polynomial function to match the given graph.

Figure 7 shows a student's solution process for matching the graph given in Figure 6. In the first frame the student has chosen a constant term of –1 and has verified that the graph of $y = -1$ defines the *y*-intercept of the given graph. The graph *near* the y-intercept is quite straight, suggesting a straight line, or linear term in the polynomial. Estimating the slope of the suggested line to be about 2, in the second frame of Figure 7 the student has chosen to add a linear term of $2x$. The resulting graph, $y = 2x - 1$, approximates the target graph very well near the y-intercept. The remaining task is to add a term which will pull the ends of the graph up steeply.



$Y=-1$

$Y=2X-1$

$Y=X^6+2X-1$

$Y=2X^6+2X-1$
Right!

Figure 7. Constructing a function for the graph presented in Figure 6, a student has proceeded, one term at a time, from the constant term to the leading term.

In the third frame of Figure 7, the student has added x^6, producing the graph of $y = x^6 + 2x - 1$, which is very close to the target graph. The sides need to be stretched even steeper, which suggests that a higher exponent or a larger coefficient is needed on the leading term. Using this diagnostic information, in the fourth frame of Figure 7, the student has matched the target graph with $y = 2x^6 + 2x - 1$. Students often used a combination of this "lowest exponent first" approach and the more standard "leading term first" approach, depending on the particular problem.

Various investigations are natural extensions of the Monomial Sums Approach. For example, ideas about turning points and *x*-intercepts can be approached by graphing a leading term and then adding successive terms to produce turning points. In order to have turning points, some terms of the polynomial need to pull the graph up, while others pull it down. Furthermore, the graph must be pulled *alternately* up and down. Hence, by manipulating the exponents and the signs of the terms in a polynomial function, it is possible to make turning points to the left of the y-axis, to the right of the y-axis, or on both sides. For example, as shown in Figure 8, the graph of $y = 4x^5 + 16x^4 + 20x^3 + 8x^2$, with alternately even and odd exponents and all positive signs, is pulled alternately up and down to the *left* of the y-axis, but it is monotone increasing to the *right* of the *y*-axis.

Given the relationship between turning points and *x*-intercepts (how many times the graph goes up and down is related to how many times it can cross the *x*-axis), this is also a good foundation for approaching Descartes' Rule of Signs: The maximum number of positive real zeros of the polynomial function $P(x)$ is the number of changes in sign of the coefficients in $P(x)$, and the maximum number of negative real zeros of $P(x)$ is the number of changes in sign of the coefficients in $P(-x)$.

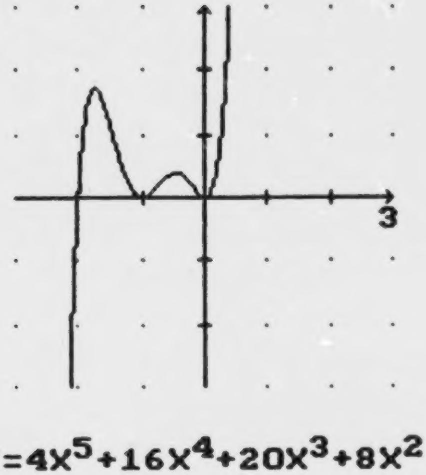


Figure 8. The graph of $y = 4x^5 + 16x^4 + 20x^3 + 8x^2$, with alternately even and odd exponents and all positive signs, is pulled alternately up and down to the *left* of the y-axis, where even-exponent terms pull it up and odd-exponent terms pull it down. However, the function is monotone increasing to the *right* of the y-axis, where all terms pull it up.

The graphical reasoning engaged in by students pursuing various investigations is important for both:

- providing a qualitative basis for understanding the behavior of polynomial functions, and
- promoting a habit of mathematical reasoning and investigation in general.

Conclusion

This paper has explored the potential of function-plotting software to facilitate mathematical reasoning and visualization of functional relationships in two contexts: trigonometric identities and polynomial functions. For both topics, new instructional approaches were designed to involve students in using graphical representations to investigate properties of functions. In contrast to the traditional procedures of rules, examples, and practice problems, the new approaches were designed to encourage:

- qualitative understanding of the behavior of functions,
- mathematical reasoning and investigation (with particular emphasis on graphical reasoning), and
- active participation in the development of mathematical ideas.

Students using the new approaches showed increased proficiency in relating functions to their graphical representations, increased reliance on mathematical reasoning, and decreased dependence on memorized rules.

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Discussion of Professor Dugdale's Paper:

Sakitani: Thank you for your very nice presentation with a beautiful voice. Now let's discuss Professor Dugdale's presentation.

Fey: Sharon, it seems to me that the difficulty you just mentioned, and you mentioned it at several points in your talk, about students somehow not ever admitting to themselves that squaring or cubing could make a number smaller is something that we've observed too. The graphical message doesn't connect to the numerical message unless you really work on it. And I wonder whether you thought at all about something comparable with tabular representation of functions. I guess it's a little harder to imagine how you'd do it, but it seems like students can almost learn to play this game with these x^2, x^3, x^4, x^5 and the graphs without having the kind of internal representation that we think they're having with it. That's more of a comment I guess than a question, but it seems to me that I think Frank's work numerical approaches to things was based on a feeling that students need to have that numerical rich experience or else the symbols and the graphs are connected in a purely abstract way that's problematic for a lot of students.

Dugdale: Yes, I certainly agree with your comment. We started with graphical exploration and used it to raise the question of why squaring or cubing a number can make it smaller. We had students resolve the apparent contradiction by numerical investigation. Each time this confusion came up, and it came up several times with the same student, we would recall the numerical investigation. Sometimes this required only a simple reminder, such as "remember what happens to one-half when you raise it to a higher power." It did seem necessary to be going back and forth between the graphical and numerical representations. The surprise to me was that we had to keep going back and forth between the two as long as we did. Of course, after a while our reminders became unnecessary. The students were noting the problem and resolving it for themselves, but it did seem that the graphical and numerical ideas needed to be connected and they had trouble staying connected. They kept becoming disconnected.

Fey: I think that's a caution to us when we use graphs, a very serious problem.

Damarin: I seems to me that that's also a place where since we're now working in increasingly computerized environments, we ought to make more specific as an objective in the earlier grades the idea that kids would learn that multiplying fractions less than one decreases rather than increases the product and it is a place where we could make very good use of drill and practice and simple game type software.

Dugdale: I agree with that and I think you have pointed out very well that a primary reason why students have this particular difficulty is that they have not encountered the idea in any definite earlier in the curriculum. I would speculate further that such ideas are absent from the lower grades because they are received as unnecessary complications, not applicable to anything immediately coming up. This may be a problem of communication between the lower grade curriculum and the upper grade curriculum as far as getting necessary concepts in place because they are needed later.

J. Wilson: This area is one in which there has been a considerable amount of research with adolescents. In their notions, in particular the misconception, that dividing by a number always gives a smaller amount. Once a misconception is learned it's awfully hard to erase it and this misconception is there because of so much drill and practice of the wrong thing. We've got to be very careful about what students are learning. Fischbein's work proposes that there is a dominance by a primitive model of division that kids learn and they learn the wrong thing and that they never have to enlarge their notion of division. They always rely on the party model for all the drill and practice and all the kinds of exercises we have in school curriculum. We need to be very careful and we need some careful research about what's going on as kids develop these ideas. I'm not a researcher at the upper elementary and the adolescent levels, but the people there need to be looking at these models and this business that we're looking at here with the graph is further evidence that the misconception stays with us a long time. Now, Turrow and her colleagues have done some work with creating cognitive dissonance situations and the kind of practice that follows that then would cause students to start questioning what they are doing and it's somehow the idea is that the dissonance needs to be pretty strong at the right level to get them to overcome the misconceptions that are habituated. I'm not disagreeing with what's been said, but I'm saying that there's a much deeper problem here and a much more pervasive problem.

Dugdale: Along with that there is also the tendency to try to avoid misconceptions or confusions by having apparent contradictions never arise and not having to deal with them. Students do not learn how to deal with what seem to be inconsistencies. They do not learn to ask the appropriate questions and make the necessary investigations to figure out what is happening. We need to foster these skills early in the curriculum and all the way through, so that students make a habit of dealing constructively with these dissonances. Of course, when a teacher is interacting with a class for only one year, it it easier to avoid these problems than to be the one who first brings them up and starts the process of learning how to deal with them.

Hashimoto: The educational considerations of the teachers is very important. For instance, in the case of the graphing problems certainly the visualization of those equations is very important but, in your case, you show the degree 5, degree 6, and very high degree polynomials. Is it really necessary? While if it were me, I would rather concentrate first on the degree 3 polynomials and give a lot of different types of 3rd degree polynomials. And then probably the students will realize that you can divide this into mainly three types/groups: $y = x^3$, $y = x^3 + x$, and $y = x^3 - x$. And then you can go on to, say, the fourth degree polynomials; then students might realize that they cannot be dividing into the different types/groups.

Dugdale: I am not sure whether this is a question of whether it is necessary to do polynomials, or a question of how to do polynomials.

Hashimoto: I'm talking about the higher degree polynomials. I said after fourth degree, but fifth, sixth too.

Dugdale: All right. We deliberately started with enough possible terms to make a lot of different combinations, because we didn't want it divided up into specific cases. The students had already done linear equations and quadratic equations and had some introduction from their textbooks to general polynomials. We wanted something general enough to require more thinking about the graphs than simply categorizing each graph as looking like x^3, $x^3 + x$, or $x^3 - x$. I think some of what Mr. Hashimoto is talking about is what we were deliberately trying not to have our students doing. Whether our way was better I don't know, but it was deliberately different.

Hashimoto: Would you repeat once again what you're really expecting from the students? What the main goal is of this teaching?

Dugdale: The main goal was to investigate how the terms of a polynomial function relate to the shape of its graph, with emphasis on the effect of each individual term being apparent in the shape of the graph. This was devoloped from the perspective of a polynomial as a sum of terms, and the graph of the polynomial as the sum of the graphs of those terms.

Hashimoto: What can you say about the relationship with the differential/integral calculus?

Dugdale: We have not yet pursued the effects of this work on students' later experience with calculus. It is certainly our hope that visualizing polynomial functions and recognizing the effect of each term will provide valuable intuitive background for treatment of polynomial functions in the calculus.

Pateman: This seems a good time to ask this: Given the capacity of the computer for both single manipulation and providing visual images, should we accept traditional content as unquestioned? And I think identities is a case in point. I'm wondering whether we should go through the gymnastics on the computer to treat that piece of traditional content or should we ask what its place is in the curriculum?

Dugdale: This question has certainly been raised. I believe the NCTM Standards advocate reduced emphasis on trigonometric identities, particularly the exercises in proving complicated identities. However, I don't know that there is much agreement that the notion of identities should be omitted; I would question that. I think that the idea of identities is still important. As a topic that traditionally involves no graphical representations, trigonometric identities provided a convenient context for comparing two approaches using graphs: one developing a graphical foundation, and the other simply adding graphing activities to a traditional approach. This work is not intended to imply that trigonometric identities should get increased attention. In fact, our graphical foundation approach did not carry proof of identities as far as the traditional course does. This de-emphasis of proving complicated identities may account for what was not a statistically significant difference, but a slight difference between the two treatments in favor of the standard treatment in having students gain facility with the ususal symbol manipulation of proving identitites.

Sakitani: I'm sorry, but we have gone over our time by 10 minutes. So, I'll stop the discussion and thank Professor Dugdale for the interesting presentation and a lively discussion.

End of discussion

Discussion of Working Group - Software Demonstration (U.S.):

Becker: We have some time left, so Professor Miwa and I thought we could use it for discussion. Are there some questions to ask Sharon, Dan or Jon? Mr. Sawada.

Sawada: For students who have played with this game, and you have had some experience with them now, what sorts of universities do they get into and do they tend to major in math?

Dugdale: I wish I could answer that. In general we have not traced students to see what universities they attend or what majors they choose. I do know, however, for the pair of students who initiated the very intricate explorations after which the rational functions part of this program is modelled: one of them dropped out of high school (laughter) and the other one became an engineering student at the University of Illinois. The student who dropped out of high school enrolled a year later in a local community college, and I don't know whether he continued beyond that level. Overall our experience suggests that we have some very bright students who are capable of doing creative things who do not find the current high school situation sufficiently motivating to keep them actively involved (even though most do not leave school). This kind of activity can help reach students who are not being reached by the regular curriculum.

Choate: I'd just like to add to that. We've used this program for years and I think it's more significant for the weaker student. We've had many students who were not interested in mathematics, really had a wonderful time with this game, and I think it really made them much more interested in mathematics. The other thing is that we had a student who just graduated from Brown University who majored in applied mathematics and he came up with probably the most creative solution that Groton School has ever seen to the expert game.

Sakitani: I've seen the software that Professor Teague was showing earlier and I could see how it would be very useful for teachers. I think there must be some real problems with giving the software to students; if you've given students the software, what happens?

Teague: Well, it's a problem for the teacher only in a limited extent. I have to think differently about how I teach. My teaching cannot be limited to techniques of solution because those solutions are available at the touch of a finger. I need to teach more about what questions to solve, where do functions come from, what function do I want to graph, not so much how to graph it. And I need to find different questions to ask to test what they know. It makes my job very, very different. And I'd like to add that we certainly don't have the answer, we don't know the best way to approach it, but we will not pretend that the software doesn't exist - it's there, it's available and we need to learn how to do it properly rather than keep it from the students.

Miwa: We are very much impressed with the software shown today. If you have any questions, we still have time for further discussion.

Choate: May I make one brief comment. Most of the software you have seen is for secondary school mathematics and beyond. There is software for elementary school mathematics in the U.S. too, but you just haven't seen any today - the people who came have more specialty at the secondary level; but don't get the impression that everything we do with software is only at the secondary level.

Uetake: We haven't seen any software you demonstrated which deals with three dimensional figures. Is there any?

Martin: There are some programs that are for more general use by lots of audiences - not just specifically for education; but there's also a program called "Visual Images" by Allen Hoffer which allows you to manipulate solids and look at cross-sections and rotations and so on.

Choate: I have a copy of 3-D Images which we are presenting to the Japanese delegation and I'll be glad to demonstrate that at some time during a break. We now have a Macintosh out in the hallway, so anyone who would like to see a program which rotates figures or which takes figures and passes them through planes, as Gary just mentioned, I'll be glad to demonstrate that for you.

Wilson: Derive, that we saw demonstrated, will do 3-D plots and Theorist which has not been demonstrated also has 3-D animation.

Demana: We also have a piece of software called Master Grapher that has 3-D option and Jim has that on his machine. That's a function plotter, it plots functions of two variables.

Sawada: One thing that I think is very useful is that in America there seem to be basically IBM and Macintosh as the two standards. In Japan we have many smaller hardwares; so, even if you develop software, it doesn't spread very widely because of the multitude of hardwares. I would like to know roughly what's the division between Mac and IBM now?

Choate: My understanding, and the university people can correct me if I'm wrong, is that the universities still are predominantly MS DOS IBM. For secondary schools there's a split between IBM and a lot of Apple IIs and that more and more schools now are beginning to use Macintoshes. If I would guess at the number of types of computers in schools and universities, I would suspect that the majority at the university level are IBM or IBM compatible, and at the secondary level either Apple II or Macintosh are the predominant machines. That's my guess.

Dugdale: I would add to what Jon says that the computers in elementary schools are still predominantly Apple II, and I expect that it will stay Apple II for quite a while, simply because when school districts have money to buy the newer machines, IBM or Macintosh, they tend to put those in the secondary schools and move the old Apple IIs into the elementary schools.

J. Wilson: The latter comment is certainly the pattern in Georgia. Apple IIs predominate and as new machines are purchased the Apple IIs are going down in the grade level. I'll just add a comment though, we find quite a few of the secondary school systems going to IBM, not only IBM, but IBM proprietary software - the whole budget goes into an IBM setup. It tends to be system-wide decisions that are made by the central administration in every case. At the university level, at our place, we have a mixture of those machines and we also have the _?_ operating system. Somehow all of this is going to tie in and for thirty years _?_ has been going to unify things for us but we're still waiting, but many of the networking situations now that we're getting into like the Sun systems that will drive remote stations has to be on a _?_ base, but you still operate, say your Macintosh or your PC, in its own language even though it's tied into the _?_ system. But there's much more coming at the university level with more intensive computing like _?_ 6000 and things like that that are on a _?_ base.

Fujii: When you want the kids to take software home to play with or to use on their own computers, you probably can't make copies for them, so what do you do?

Wilson: In some cases, depending on the software, you'd purchase a site license which allows you to make copies. I think Sunburst has a pack they sell in which you have, say, 30 copies that you can have in a library; but certainly there are concerns of copyright that must be adhered to and in the software that we use most widely is site-license purchase.

Demana: The software we developed called Master Grapher, we've been able to keep the price extremely low by marketing through Addison-Wesley. For example, it goes free with our textbooks. Site licenses are $300 and in those cases students can make copies and take them home or use at school, so there are some softwares like that which you can keep and if you battle the publishers, the publishers want to charge an arm and a leg, but you can keep the price down and use site licenses and allow students to take them home.

Uetake: For the software we've seen today, it doesn't look like the sort of thing you can use immediately with a glance at the manual. It seems as if it would require a certain amount of teacher education and training. Would anyone care to speak to that?

Choate: I think I presented probably the most far out piece of software. In fact, I suspect there are members of the American delegation who haven't seen it. The training comes from taking courses at conferences. A few colleges have courses in how to use that type of software in the classroom. I think the predominant way that someone would learn about it would be to go to one of the National Council of Teachers of Mathematics conferences or several other conferences around the country, that show people how to use it.

Demana: Most software have a manual that goes along with it that details very carefully how to use it and most of the time how to use it to do mathematics or whatever it's spelled out to do. And sometimes there's a package which includes lesson plans, so there is that ability too; for example, with Master Grapher we have plenty of detailed instructions of how to not only graph functions in three dimensions but even mathematical examples of how to use it in a classroom.

Teague: I think that the more flexible the software, the more different things it does, the harder it is to start using. Some software we've seen does one thing, rolling a rectangle for example; it does one thing and so it's fairly easy for a teacher to pick it up and hit the space bar and go from one thing to the next. Software like Derive that does many, many different things is going to be harder to learn to use because there are many more commands to learn.

Martin: I would just add that I think especially with a lot of the programs designed on the Macintosh, they are designed to be open-ended and very fluid so you can begin playing with them very quickly; for example, with the Sketchpad I just tried things and learned. Eventually I got around to reading the manual to see if there were things that I was missing but, in general, with the Macintosh programs you can begin working very quickly because of the standard interface. I think the bigger issue than that is in some sense instilling a philosophy or helping teachers to see the rationale and how a program can impact with what they're doing.

Kumagai: I have two questions. First of all, what sorts of people are developing software? And second, is there any sort of a rating or evaluation organization for all the software?

Demana: I'm not sure I can answer all the questions, but there are reviews put in the Mathematical Association of America publication called The Notices which has a column on computers and evaluates software regularly with each issue. There's a college mathematics journal that does the same thing, has a column and reviews software on a regular basis. Software developers cut across the range of spectrums from school teachers to formal companies that try to put it together.

Damarin: At the level of kindergarten through twelfth grade, for elementary, secondary, and high school software, there are a couple of organizations, one publicly funded and at least more than one privately funded that can hire reviewers. Probably the most famous of those is APCE, which is a private organization and one can acquire software reviews over a computer based bulletin board - you can interface through your modum and get software reviews from them. Many teacher magazines also publish software reviews. Parents magazine publishes software reviews and, in fact, gives awards for software. And then there's an organization called Only The Best which takes all of those reviews and creates a list of only the best. So, I guess the main message is there's quite a software review endeavor. Even so, each school district feels the need to do its own software review. So it's quite a complex.

Sawada: In Japan when you're selling software to teachers, what happens often is that teachers use it two or three times and then get tired of it and want to start modifying it themselves. This may be more difficult in the United States, but is there a development of software which is essentially customizable, which can be changed in small parts?

Choate: I think this brings up a major issue. A lot of the software in the United States is software that allows you to do a lot of different things. Let me give a very specific example: if I wanted to do the lesson with a rotating rectangle which we have seen, I would do that using the geometer's Sketchpad. And I think if you look at the software used in the United States now, we've developed a lot of tools with a lot of flexibility that in a sense allows the teacher to build the lessons. I think what's happened is the tools are getting easier and easier to use because if you look at American textbooks and go back about six years, they used to have included in them lines of BASIC programming language. They don't have those anymore and now instead of referring to a programming language they refer to a tool like the Geometric Supposer or Green Globs or the Geometer's Sketchpad or Derive. And I think what you'll probably see is the development of these tools, once you have an idea of what you need in a classroom for achieving this sort of open-endedness that you'd like for the teachers to take and change. And what will happen is people will start developing modules within a tool that they can then change.

Miwa: I am very sorry to have to stop this session. We have had very useful discussion. Thank you for your cooperation.

End of Discussion of Working Group

Mathematical Visualization in Problem Solving Facilitated by Computers

James W. Wilson

University of Georgia

I wish to recognize the assistance of Earl Bennett, David Barnes, Simon Hau, and Maria Fernandez of the University of Georgia and Nelda Hadaway from Georgia State University in preparing material in support of this paper.

Mathematical Visualization in Problem Solving Facilitated by Computers

> *Your problem may be modest; but if it challenges your curiosity and brings into play your inventive faculties, and if you solve it by your own means, you may experience the tension and enjoy the triumph of discovery. Such experiences at a susceptible age may create a taste for mental work and leave their imprint on mind and character for a lifetime.* (Polya, 1944, p. v.)

Using Computers

There are many uses of computers in doing mathematics, teaching mathematics, and learning mathematics. Computers and the software for them are marvelous tools. And just as I would have a variety of tools in my workshop for various tasks, I want an array of software, computers, and calculators available to select from, as appropriate, for a particular task.

The variety of uses includes

- preparation of instructional materials with word processors, sketching programs
- writing computer programs for problem solving
- classroom demonstrations
- mathematical investigations (for teacher or student)
- student uses during instruction
- student use for assignments/homework
- recreation and games
- record keeping
- using applications
- drill and practice

In my presentation, I will concentrate on activities that might be used for classroom demonstration or for mathematical investigations.

Mathematical Visualization

The ability to visualize mathematical relationships is an essential part of many people's knowledge of mathematics and their facility in communicating ideas about mathematics. Computers enable us to extend this capability. In my presentation I will discuss some activities at the University of Georgia as examples of using computers in the teaching of mathematics or in using computers as a tool to explore problem situations. These activities include classes I teach on mathematical problem solving for secondary teachers, classes for prospective elementary teachers, work with inservice teachers (who in turn may be using computers in their classes), work with graduate students who are engaged in teaching and research, materials development projects, and prototype planning with engineers from Georgia Institute of Technology.

I have provided several background papers. One is a research review by Earl Bennett. Others are problems presented from explorations which I might like to show here but will not have the time to do so. My presentation will concentrate on example problems. Some of this material will be part of a forthcoming synthesis of problem solving research by Wilson, Hadaway, and Fernandez.

Problem Solving

It is useful to have a framework to think about the processes involved in mathematical problem solving. Most formulations of a problem solving framework in U. S. textbooks attribute some relationship to Polya's problem solving stages (1945). These stages were described by 1) understanding the problem, 2) making a plan, 3) carrying out the plan, and 4) looking back.

Polya also stated that problem solving was a major theme of **doing mathematics** and when he wrote about what he expected of students, he used the language of "teaching students to think" (1965). "How to think" is a theme that underlies much of genuine inquiry and problem solving in mathematics. Unfortunately, much of the well-intended efforts of teaching students "how to think" in mathematics problem solving gets transformed into teaching "what to think" or "what to do." This is, in particular, a byproduct of an emphasis on procedural knowledge about problem solving such as we have in the linear framework of U. S. mathematics textbooks and the very limited problems/exercises included in lessons.

> *Thus a teacher of mathematics has a great opportunity. If he fills his allotted time with drilling his students in routine operations he kills their interest, hampers their intellectual development, and misuses his opportunity. But if he challenges the curiosity of his students by setting them problems proportionate to their knowledge, and helps them to solve their problems with stimulating questions, he may give them a taste for, and some means of, independent thinking.* (Polya, 1944, p. v.)

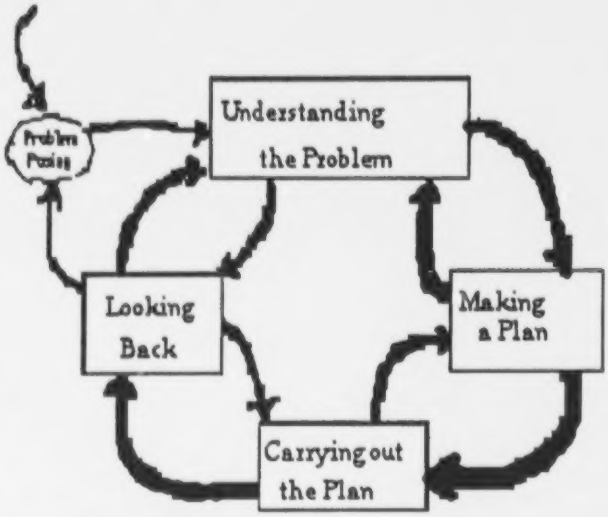
Clearly, the linear nature of the models used in numerous textbooks does not promote the spirit of Polya's stages and his goal of teaching students to think. By their nature, all these traditional models have the following defects:

1. They depict problem solving as a linear process.
2. They present problem solving as a series of steps.
3. They imply that solving mathematics problems is a procedure to be memorized, practiced, and habituated.
4. They lead to an emphasis on answer getting.

These linear formulations are not very consistent with genuine problem solving activity. They may be consistent with how experienced problem solvers may present their solution and answer after the problem solving is completed. In an analogous way, mathematicians present their proofs in very concise terms, but the most elegant of proofs may fail to convey the dynamic inquiry that went on in finding the proof.

There is a dynamic and cyclic nature of genuine problem solving. A student may begin with a problem and engage in thought and activity to understand it. The student attempts to make a plan and in the process may discover a need to understand the problem better. Or when a plan has been formed, the student may attempt to carry it out and be unable to do so. The next activity may be attempting to make a new plan, or going back to develop a new understanding of the problem, or posing a new (possibly related) problem to work on.

The following framework is useful for illustrating the dynamic, cyclic interpretation of Polya's stages. It has been used in my mathematics problem solving course for many years.



Any of the arrows could describe student activity (thought) in the process of solving mathematics problems. Clearly, genuine problem experience in mathematics can not be captured by the outer, one-directional arrows alone. It is not a theoretical model. Rather, it is a framework for discussing various pedagogical, curricular, instructional, and learning issues involved with the goals of mathematical problem solving in our schools.

What is mathematics?

If our answer to this question uses words like explore, inquiry, discover, plausible reasoning, or problem solving, then we are attending to the processes of mathematics. Most of us would also make a content list like algebra, geometry, number, probability, statistics, or calculus. Deep down, our answers to such questions as: What is mathematics? What do mathematicians do? What do mathematics students do? Should the activities for mathematics students model what mathematicians do? can affect how we approach mathematics problems and how we teach mathematics. Moreover, the answer to the title question "What is mathematics?" is not necessarily the same as our answer to the question "What is school mathematics?"

> *Many a guess has turned out to be wrong but nevertheless useful in leading to a better one.* (Polya, 1957, p. 99)

The art of problem solving is the heart of mathematics. What a pity that so many students never experience mathematics is this way. The NCTM recommendations (1980, 1989) to make problem solving the focus of school mathematics posed fundamental questions about the nature of school mathematics.

Why Problem Solving?

The NCTM (1980,1989) has strongly endorsed the inclusion on problem solving in school mathematics. There are many reasons for doing this. First, problem solving is a major part of mathematics. It is the sum and substance of our discipline and to reduce the discipline to a set of exercises and skills devoid of problem solving is misrepresenting mathematics as a discipline and shortchanging the students. Second, mathematics has many applications and often those application represent important problems in mathematics. Our subject is used in the work, understanding, and communication within other disciplines. Third, there is an intrinsic motivation embedded in solving mathematics problems. Thus, we include problem solving in school mathematics because in can stimulate the interest and enthusiasm of the students. Fourth, problem solving can be fun. Thus many of us do mathematics problems for recreation. Finally, problem solving must be in the school mathematics curriculum to allow students to develop the art of problem solving. This art is so essential to really understanding mathematics and appreciating mathematics that it must be an instructional goal.

Teachers often give strong rationale for not including problem solving activities is school mathematics instruction. These include arguments that problem solving is too difficult, problem solving takes too much time, the school curriculum is very full and there is no room for problem solving, problem solving will not be measured and tested, mathematics is sequential and students must master facts, procedures, and algorithms, appropriate mathematics problems are not available, problem solving is not in the textbook, and learning procedural mathematics should be emphasized in school mathematics so that problem solving can be done later. There are some teachers and curriculum leaders who argue that drill and practice with the basics is the first priority of school mathematics, and since many students have not mastered the basic facts, problem solving should not be attempted. Finally, there is the deceptive practice of claiming to emphasize problem solving when in fact the only emphasis is on routine exercises.

Some University of Georgia Activities

Project LITMUS

Project LITMUS is directed by Larry Hatfield and funded by the National Science Foundation to provide total, district-wide infusion of technology in two Georgia School System. Over the next five years, Dr. Hatfield and the Project LITMUS will work with all teachers in the two districts to develop facility with the use of calculators and computers as tools for mathematics teaching and learning.

This summer, 47 teachers (some elementary, some middle school, some secondary) are receiving intensive exposure to computer and calculator use. These 47 teachers are the Leader Teachers in the district and will each have two computers (a MacIntosh LC and a MacIntosh Classic, plus classroom sets of calculators, etc. to use over the next year and develop their expertise as they use technology to some extent with their students. In 1992 - 93 and beyond these leader teacher will have additional training and also have a role in the training of peer teachers in their schools and will assist with the incorporation of technology into work with students in their individual schools. The school system has and the University of Georgia has made a substantial investment in computers and calculators since these are not funded by NSF.

The philosophy of LITMUS is rooted in using technology as tools to enhance instruction that has a strong problem solving component. The elementary level is using some LOGO microworlds developed by John Olive, but for the most part units are being built around tool software applications such as GSP, Graph Wiz, Basic, Object LOGO, Smartworks, Excel, etc.

Christopher Columbus Consortium

The Christopher Columbus Consortium is a national "network" of schools and universities who have been prodded into collaboration by an equipment grant from Apple Foundation. The University of Georgia is matched with a local high school, Clarke Central High School, and, in particular, Three mathematics education faculty, two doctoral students, and nine high school faculty have formed a team to implement the use of MacIntosh computers into mathematics instruction. There are demonstration computers that reside in the classrooms of nine of the faculty and there is an "open access" laboratory where students can go for individual work on the computer or where classes can be taken for group instruction. In 1990-91, instructional activities with computers took place in geometry classes using GSP, in algebra classes using Graph Wiz and the Mathematics Teachers Workstation, and in calculus classes using Theorist.

The purpose of our work in the Christopher Columbus Project is threefold. First, we wish to develop a framework to introduce teachers to new technology and new capabilities of the technology as it becomes available. The use of technology by mathematics teachers has been stymied by a stereotype of using computers to deliver instruction. This is an unfortunate impact of work in CAI. We feel the framework of mathematical investigations is an orientation that will bring about much more potent use of the computer as a tool for exploring and problem solving. We want to develop a mentality of "what if" and an attitude that we can use the computer or calculator to find out, make new hypotheses, and construct mathematical ideas. Second, we want to make teachers comfortable with the appropriate use of computers and calculators -- to make these as comfortable and as ubiquitous as the overhead projector. Our primary focus is on the teacher's use of the technology to demonstrate and explain. Individual student use of the technology is a natural extension when made a part of overall instructional design. Third, we want to nourish collaboration between the Department of Mathematics in the school and the Department of Mathematics Education of the university.

Specifically, goals for consortium will include the following:

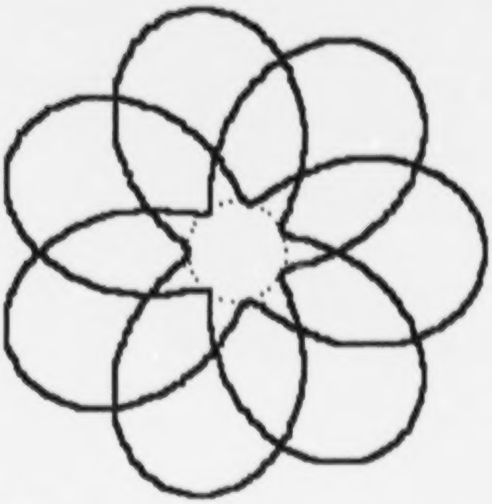
- Preparation of mathematical investigations stimulated by topics in secondary school mathematics.
- Cooperative research on the teaching and learning of mathematics in the presence of computer tools.
- Preparation of illustrations and demonstrations of the use of various software and technology in mathematical investigations.

Mathematics teachers have little time for digging things out on their own as new technologies are made available. They need models. Unfortunately, many of the models of technology use are still throwbacks to stereotypic CAI -- driven by the mentality that computers can deliver instruction. The MacIntosh and many of the software programs recently made available on it, and the TI-81 calculator, bring about some new capabilities of the technology. Among these capabilities are rapid graphics, dynamic programs, animation, and simulation.

Rapid graphics are important in classroom demonstrations. If the drawing of a curve takes more that a few seconds the teacher may be faced with problems of keeping a class engaged. If it takes more than a few seconds then there is little likelihood of examining several examples with changing parameters. Recent programs such as the Mathematics Teachers Workstation, Graph Wiz, Geometer's Sketch Pad, or Theorist can meet this requirement running on current machines (e.g. SE, SE/30, LC, II, IIcx, IIsi). For example, multiple graphs such as

$$r = 3 + 3\cos(nt)$$

can be examined for $n = 3, 4, 5, 6$ in rapid succession. The graph for $n = 15$ can be drawn in a matter of seconds to verify the student's prediction of the number of "loops" or "petals." It can also be drawn for $n = 3/5$ -- bringing into question most textbooks label of the "n-leaf rose" -- or $n = 1/2, 1/3, 1/4 \ldots$ Or, what about changing the 3 coefficient of the cos(at) term and produce a graph like this:



R(t) = 3 + 2cos((7/3) t)

And, the students, or the teacher, can contrast all these results with

$$r = 3 + 3\sin(nt)$$

or

$$r = 3\cos(nt)$$

or

$$r = 3\sin(nt)$$

Teachers need to understand the capabilities of different software programs. For example, the investigation above would work well with the Mathematics Teacher Workstation (MTW) or Theorist. A menu in the MTW makes available the graphing of polar equations and the parameters can be set to allow any number of revolutions in drawing a graph (for n an integer, one revolution is needed; for n =a/b, b revolutions are needed (a,b) =1). Graph Wiz, on the other hand, could not be used to make these graphs for non-integer n.

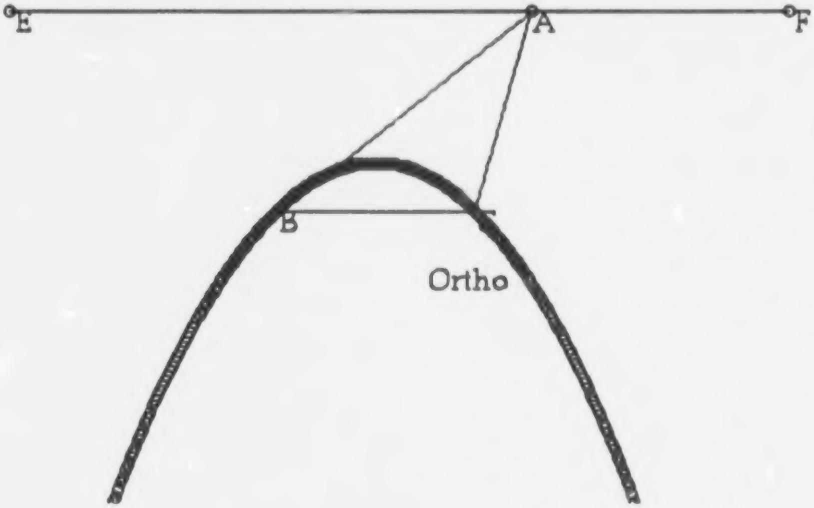
The Geometric Supposer (triangles) and the Geometer's Sketch Pad allow dynamic investigations. For example, with the Supposer, the teacher might draw a triangle and its three medians. There is opportunity for discussion of the observation that the three medians of this triangle intersect in a single point. Then measure of each median and the distance of the intersection from the opposite side can be obtained and the ratio computed. Again, discussion can focus on the result that the ratio is 1/3 for each of the three medians. Now, the dynamic aspect is in the "Repeat" function. The software allows using the mouse to relocate the vertices of the triangle (1, 2, or all 3) and repeating the construction and measurement in a new triangle. And another. And then the students can be led to investigating particular triangles such as isosceles or right triangles.

The Geometer's Sketch Pad is less user friendly but has many more capabilities. For example, the program can be used to construct the orthocenter, centroid, incenter, and circumcenter in the same triangle. What happens to these four centers as the shape of the triangle is changed? The mouse can be used to move a vertex and all of the other relationships in the drawing change with it. Thus if all triangles on a fixed base having the same area are examined (e.g. move the third vertex along a line parallel to the base) all of the following can be observed and conjectured:

. . . the circumcenter moves in a locus orthogonal to the base . . .

. . . the centroid moves along a line parallel to the base . . . and . . .

. . . the orthocenter moves along a locus that is a parabola that goes thru . . .

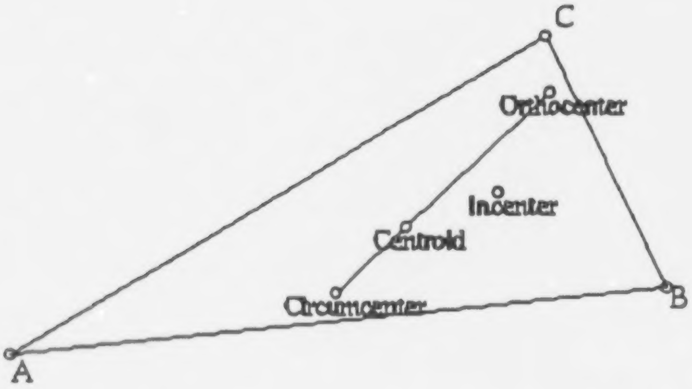


. . . the incenter has a strange looking curved locus . . .

. . . the circumcenter, centroid, and orthocenter are always colinear . .

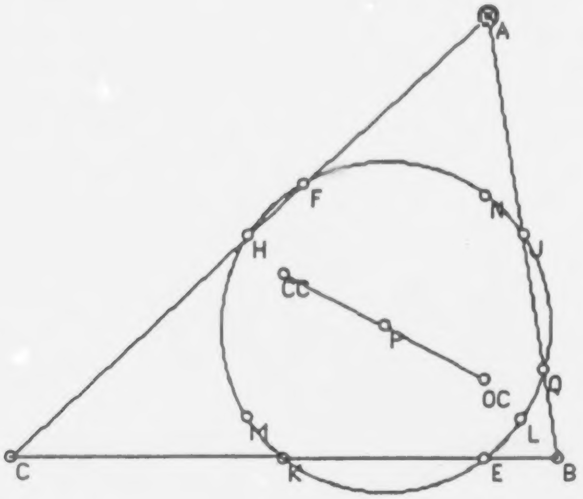
. . . the centroid is always between the other two . . . except when . . .

. . . the distance from the centroid to the orthocenter is always twice the distance from the centroid to the circumcenter . . .


C
Orthocenter
Incenter
Centroid
Circumcenter
B
A

. . . the line segment from the orthocenter to the circumcenter does a "flip" about the centroid as the third vertex is moved from side to side . . .

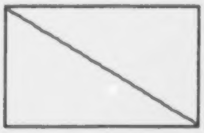
. . . the center of the nine-point circle . . .


A
F
N
J
H
CC
P
OC
Q
M
L
C
K
E
B

This investigation can be set up and done in a few minutes. All of the computations and drawings are instantaneous. This is, of course, the Euler line of a triangle and its investigation without technology has been outside the realm of elementary geometry. The Geometer's Sketch Pad has great potential for teachers and students to use in investigations of ideas in geometry.

- Given 100 feet of fencing, ...
  a. What is the maximum area that could be enclosed in a rectangular region?
  b. What is the maximum area that could be enclosed in a triangular region?
  c. What area could be enclosed in a regular hexagon? regular octagon?
  d. What area could be enclosed in a regular n-gon? Explore ... What happens as n increases?
  e. Show that the square has maximum area or any quadrilateral with a perimeter of 100.
  f. Find five triangle with perimeter of 100 having integer sides and integer area.
  g. Find the area of a quarter circle region having a perimeter of 100.
  h. Find the area of a semicircle region having a perimeter of 100.
  i. What region bounded by two radii and the arc of a circle having a perimeter of 100 will have maximum area? No calculus!
  j. ETC.

This investigation can continue to grow as students begin to pose their own extensions of the problems. The whole set of problems of building a pen with the fencing using some natural boundary can be posed. (Standard fare in calculus, but better here). Another set of extensions would be to maximize the area of pens with 100 feet of fencing that includes one or more partitions. For example:



Other WINGZ investigations might flow from simulations around data bases for a budget, car ownership, income tax, or a business.

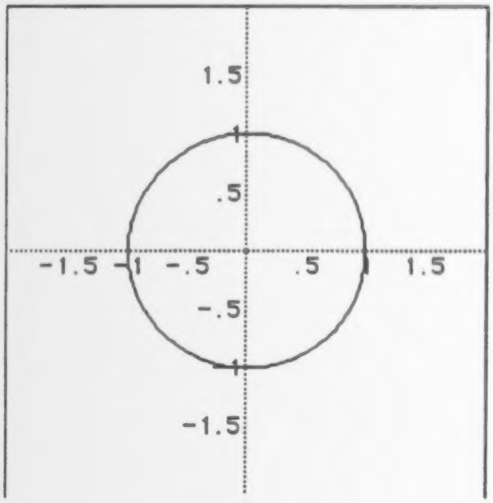
Graph Wiz is an application for graphing relations developed by Alan Hoffer. Most graphing programs require transforming the relation you want to graph to the form $y = f(x)$, which may be somewhat cumbersome. Graph Wiz on the other hand constructs the graph from whatever form the relation is in. For example, the teacher or students might investigate the following graphs by entering these equations exactly in the form given here:

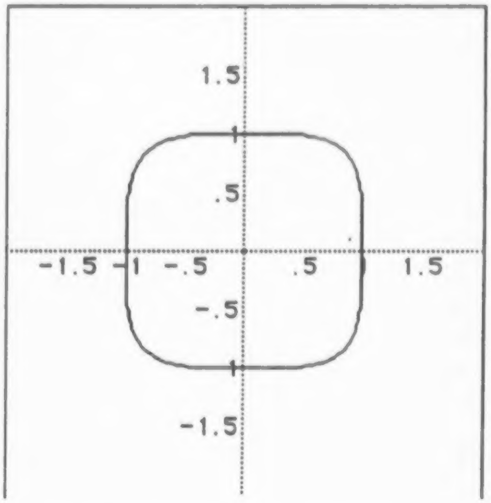
$$x^2 + y^2 = 1$$

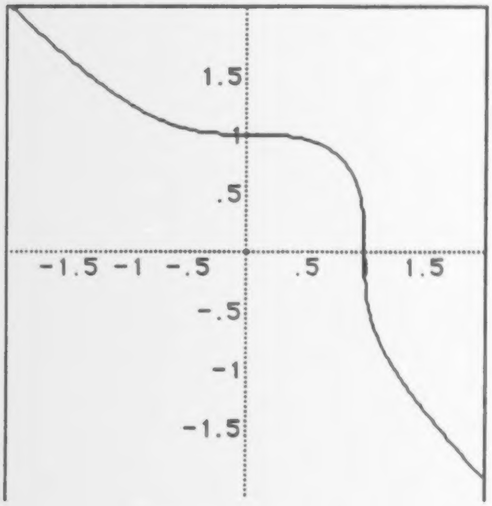
$$x^3 + y^3 = 1$$

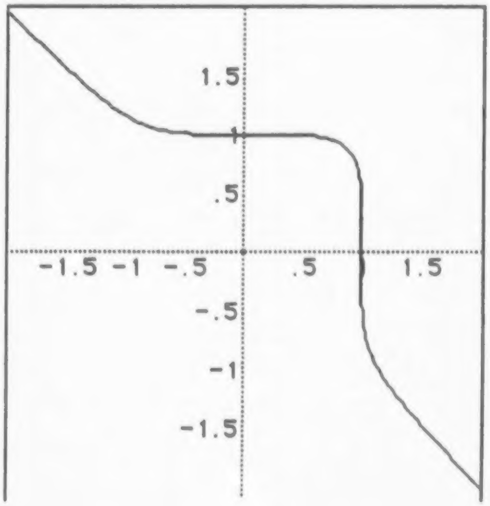
$$x^4 + y^4 = 1$$

$$x^5 + y^5 = 1$$



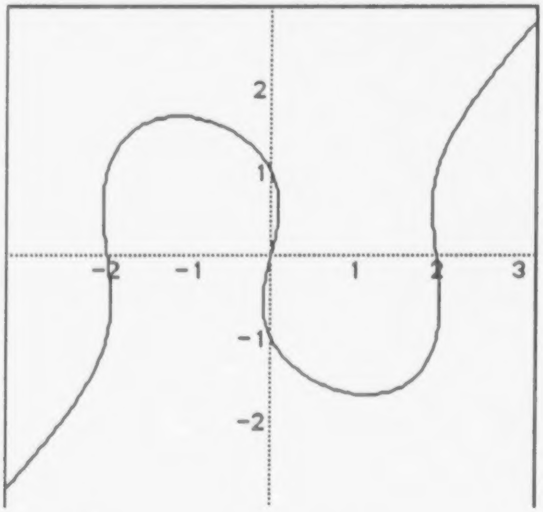






Seeing these graphs, either singly or concurrently on the same axes, can not help but generate some guesses for further confirmation. Other software would not be useful to investigate these relations in "real time" -- that is, say, in the context of an ongoing class. Graph Wiz also has read-out tools to determine coordinates wherever the cursor is placed and it has tools for graphing inequalities.

The Curve Building demonstration paper was done using Graph Wiz to explore the relation $(y^2 - 1)y = (x^2 - 4)x$:



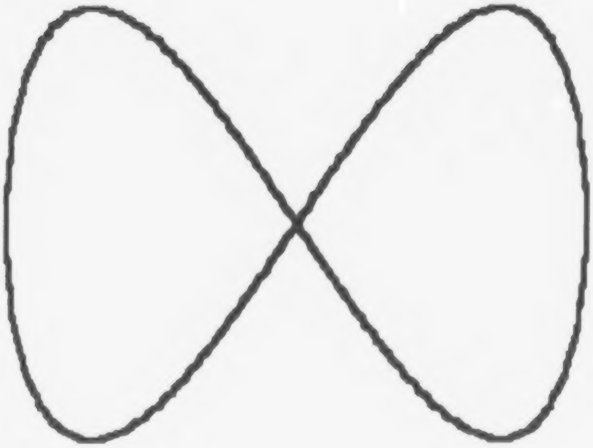
Mathematics Teacher Workstation was developed by Ray Carry. It began development as a function grapher to handle rectangular, polar, and parametric equations. The Geometric Construction kit was added to give on-screen tools to simulate ruler and compass constructions. Features were added to provide retrieval and sequencing of images from the uses of the program. Its parametric equations grapher is more user friendly than any other software available. The graph in the demonstration problem on using parametric equations was done with MTW.

One set of curves for exploration uses the following parametric equations.

$$x(t) = 4\sin((a/b)t)$$

$$y(t) = 3\sin(t)$$

various curves can be generated for different a and b, $b \neq 0$. For example, $a = 1$ and $b = 2$ gives the following



x(t) = 4sin((1/2) t)
y(t) = 3sin(t)

Theorist is a very powerful graphics display and symbol manipulator program. Its power is also a limitation because setting up most investigations would take too much time for in-class demonstration. It will, however, show animations and three dimensional images. Prior preparation and set-up can be saved in a "notebook" (Theorist's word for "file") and then played back for class time demonstrations. For example, one animation demonstrated at Clarke Central this Fall was to take a curve y = sin(x) and animate a tangent line moving along the curve. The slope of the tangent line was plotted as a trace, producing y = cos(x) . . . This animation can be played back at various speeds, stopped at any point for illustration and discussion, or extended to other functions.

Georgia Institute of Technology Planning

As Atlanta prepared its presentation to the International Olympic Committee meetings in Tokyo, the Georgia Institute of Technology staff were involved in a multimedia presentation built around a model of the proposed Olympic Village. The presentation was dramatic and many feel it was crucial in the successful bid for the 1996 Olympic Games. It involved interactive videodisc, interactive computer programs, and carefully prepared script. Subsequently, discussions were initiated with University of Georgia faculty in mathematics education, Georgia Institute of Technology engineering, mathematics, and architectural design, and technology staff from various industries as to whether this multimedia technology and the coming fiber optic cable network could be used in meaningful ways for education. In particular, the group was looking for innovative classrooms and innovative curriculum. Discussions continue. We are discussing innovations that might be available in 1996 or later.

The hardware proposals include development of student datapad/consoles with video quality images. A classroom, or a group of classrooms, my be powered by a computer of the RISC 6000 class with an extensive library available to the teacher console. The teachers console could be divided into up to 16 different to monitor the work of individual or groups of students. The fiber optic network would make possible interactive 2-way TV to remote sites. Some prototypes have been assembled by the Georgia Tech and industry engineers.

The courseware proposals, at this time, are moving toward proposals for 10 to 20 "modules" -- units of instruction -- at the secondary level that could be targeted toward mathematical visualization. There are other subject materials under discussion -- such as science units built from archival material of the National Geographic. Videodisc materials for adult literacy instruction -- teaching reading to adults -- are also under discuss as well as delivery of instruction to non-school sites such as homebound students or to prisons. The current outline of module ideas is listed in the Appendix 1.

Research Activities

Other than to note the interest and the activity, I will not delineate the research that is proposed or under way. Several faculty and graduate students are engaged in LITMUS and the Christopher Columbus Consortium. We have studies underway and proposed to study the schools, the teachers, and the students. The review by Bennett is part of an effort to organize studies of how mathematical visualization capabilities develop in the presence of various technology.

Materials

Appendix 2 has additional problems and materials I have used with computer exploration

APPENDIX 1
Outline of Module Topics

I. PROPERTIES OF TRIANGLES

1. Circumcenter/ Perpendicular Bisector of Sides
2. Incenter/ Angle Bisectors (and excenters)
3. Centroid/Medians
4. Orthocenter/Altitudes
5. Other Cevians
6. Euler line
7. Loci problems
8. Nine-Point Circle
9. Pythagorean Relations

II. SIMILARITY

1. Center of Similarity; projections
2. Similarity coefficient
3. 3-D
4. 2-D
5. Dilation ..... see Video Tape from Project Mathematica . . .

III. CONGRUENCE

1. Polygons
2. Polyhedra
3. Etc.

IV. CIRCLES AND SPHERES

1. Arcs
2. Central angle relationships
3. Chords, Secants, Tangents
4. Intersecting circles
5. Great circles

V. DISTANCE, AREA, AND VOLUME

1. Concepts
2. Formulas
3. Applications
4. Maximization (Minimization) Problems
5. Isoperimetric inequalities and relationships 2-D and 3-D
6. Heron's Formula
7. Brahmagupta's formula
8. Visualization of 4 dimensions

VI. TRANSFORMATIONAL GEOMETRY

1. Basic isometries
2. Coordinatization
3. 3-D translations, rotations

VII. PROJECTIVE GEOMETRY

1. Cross Ratio
2. Desargue's Theorem
3. Pappus's Theorem
4. Pascal's Theorem Pascal Line of a hexagon inscribed in a circle.
5. Menelaus's Theorem
6. Duality
    - . . . principle
    - . . . examples
7. Dual point/line for conics
8. Perspective drawings
9. Drafting basics ... e.g. Mechanical Drawing, perspective

VIII. CONIC SECTIONS

1. 3-D models; 3-D images
    - . . . Intersection of Plane and double cone
2. Projection of a ring
    - . . . circle
    - . . . ellipse   See Nicollett Films
    - . . . parabola
    - . . . hyperbola
3. Animation: directrix and focus
4. Other animation
5. Paper folding
6. Eccentricity coefficient
7. Analytic geometry

. . . Formulas
. . . xy coordinates
    ... graphs with center at origin
    ... other
. . . polar equations
. . . parametric equations

IX. POLYGONS AND POLYHEDRA

1. Quadrilaterals
2. Polygons and Regular Polygons
3. Euler's Formulas
4. Platonic Solids
5. Archimedian Solids
6. Model building; nets
7. Projections
8. Stellated Polyhedra

X. TRIGONOMETRIC RELATIONSHIPS

1. Basic concepts
2. Graphs
3. Polar coordinates
4. Complex numbers
5. Applications

XI. LINEAR ALGEBRA

1. 2-D
2. 3-D

n. Non-linear systems

XII. ITERATION AND RECURSION

1. Iteration of pattern
2. Iteration of function
3. Use of iterations to find roots
4. Recursive functions
5. Fractals

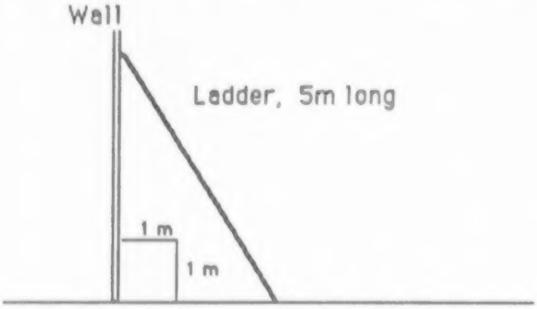
XIII. GRAPHS

1. Functions
2. Relations
3. Discontinuities
4. Composite graphs
    - $f(x) + g(x)$
    - $f(x)\,g(x)$
    - $f(g(x))$
    - $x = g(x)$ obtained from $f(x) = 0$
5. Explorations with Polar equations
6. What if? activities

APPENDIX 2

Additional Problems for Computer Exploration

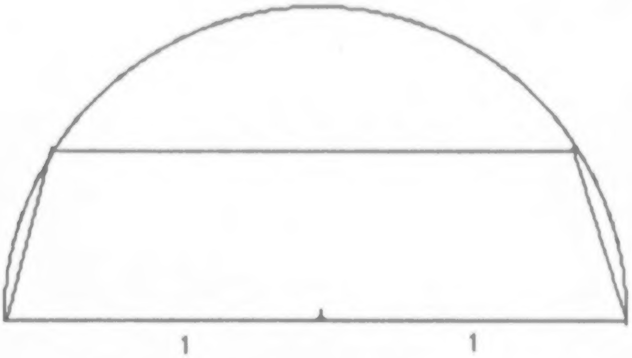
1. A ladder 5 meters long leans against a wall, reaching over the top of a box that is 1 meter on each side. The box is against the wall. What is the maximum height on the wall that the ladder can reach? The side view is:



Assume the wall is perpendicular to the floor. Use your calculator to find the maximum height to the nearest .01 meter.

2. How long is the groove on one side of a long-play (33 1/3 rpm) phonograph record? Assume there is a single recording and the Outer (beginning) groove is 5.75 inches from the center and the Inner (ending) groove is 1.75 inches from the center. The recording plays for 23 minutes.

3. Find the maximum area of a trapezoid inscribed in a semicircle of radius 1.



Hint: Use the arithmetic mean-geometric mean inequality

4. Exploration
   i. Graph $x^3 + y^3 = 3axy$ for various values of a.

   ii. Graph $x^3 + y^3 + b = 3axy$ for various values of a and b.

   iii. Examine the graphs of

$$(\sin(x))^3 + (\sin(y))^3 = 3(a)(\sin(x))(\sin(y))$$

   for $a = 1/2, 5/8, 3/4$.

   iv. Examine the graphs of

$$(\tan(x))^3 + (\tan(y))^3 = 3(a)(\tan(x))(\tan(y))$$

   for a range of values for a. What happens when a constant is added to the left hand side?

CURVE BUILDING

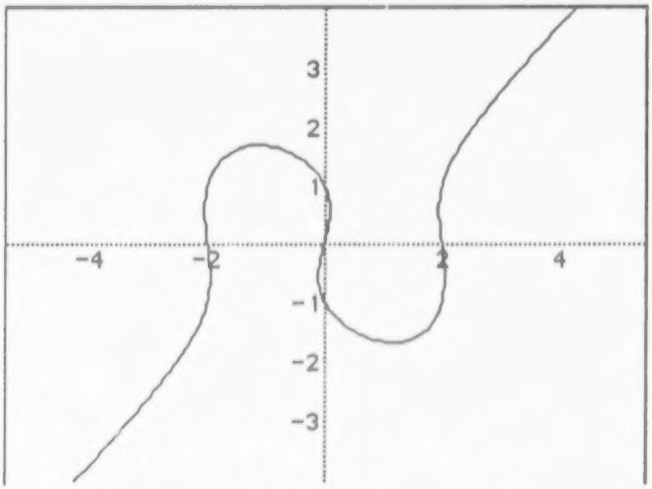
James W. Wilson

This exploration with graphs of mathematical relations grew out of some class discussions with prospective teachers as we examined the use of the computer for graphing mathematical relations. The use of the computer allows us to explore and conjecture and develop ideas with respect to the mathematical relations and their graphs.

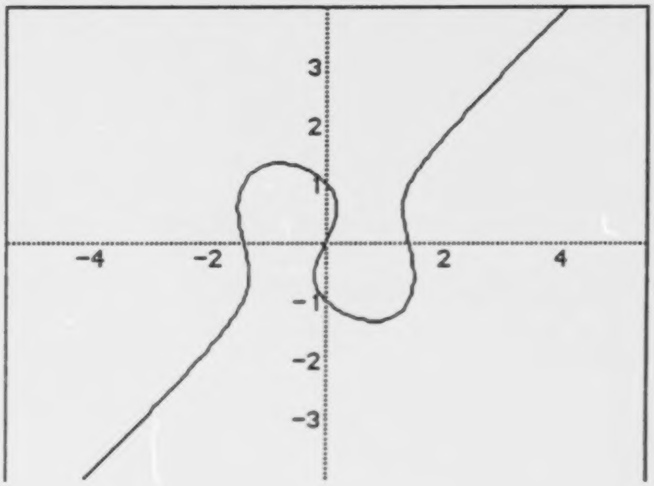
The particular relation was

$$(y^2 - 1)y = (x^2 - 4)x$$

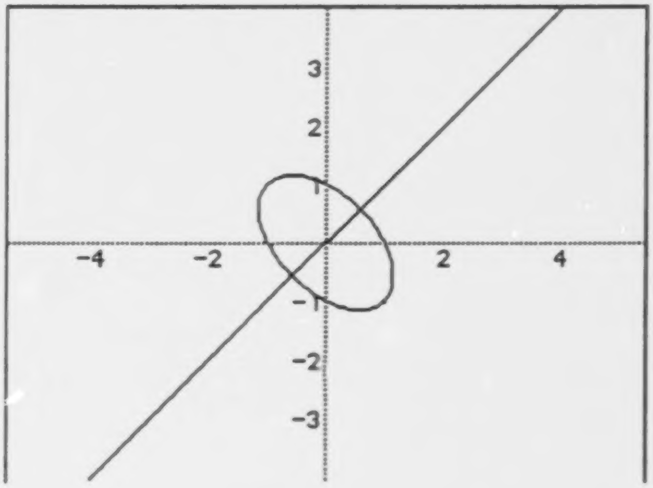
Cursory discussion led to agreement that the curve would cross the y-axis at three points -- -1, 0, and 1 -- and likewise the curve would cross the x-axis at three points -- -2, 0, and 2. Also, for large x, y is large, etc. The computer provides a tool to draw the graph, look closely at particular regions of the graph, and so forth. It was clear that the interesting part of the graph would be near the origin. Using Graph Wiz (Hoffer, 1989) the graph was obtained as follows.



A point of discussion was to explore changes in the graph that resulted from changes in the relation. For example, the following resulted when the 4 on the right hand side was change to 2. The curve as a somewhat similar shape but is more compressed, crossing the x-axis at points closer to the origin.



Why not go further and see what happens when the constant on the right hand side is 1, producing a relation symmetric with respect to x and y? Surely the symmetry will add some demand to the graph. The result is the following.



$$\left(y^2 - 1\right)y = \left(x^2 - 1\right)x$$

The graph now looks like the composite of two graphs -- a line and an ellipse. It is a single relation, but some algebra -- either by hand or by symbol manipulator -- can lead to a factored form of the equation,

$$(x - y)\left(x^2 + xy + y^2 - 1\right) = 0$$

and one factor corresponds to the line and the other corresponds to the ellipse.

Why not reverse this process? Let's begin with two mathematical relations and consider their product to build a graph. Then coefficients or constant terms in the product can be changed to build related curves.

Take the ellipse, express its equation, and graph it. For example,

$$y^2 + xy + x^2 = 4$$

is an ellipse with major axis along the line $y = -x$ and contains the points (0,2), (0,-2), (2,0), and (-2,0). A graph is in Figure 1.

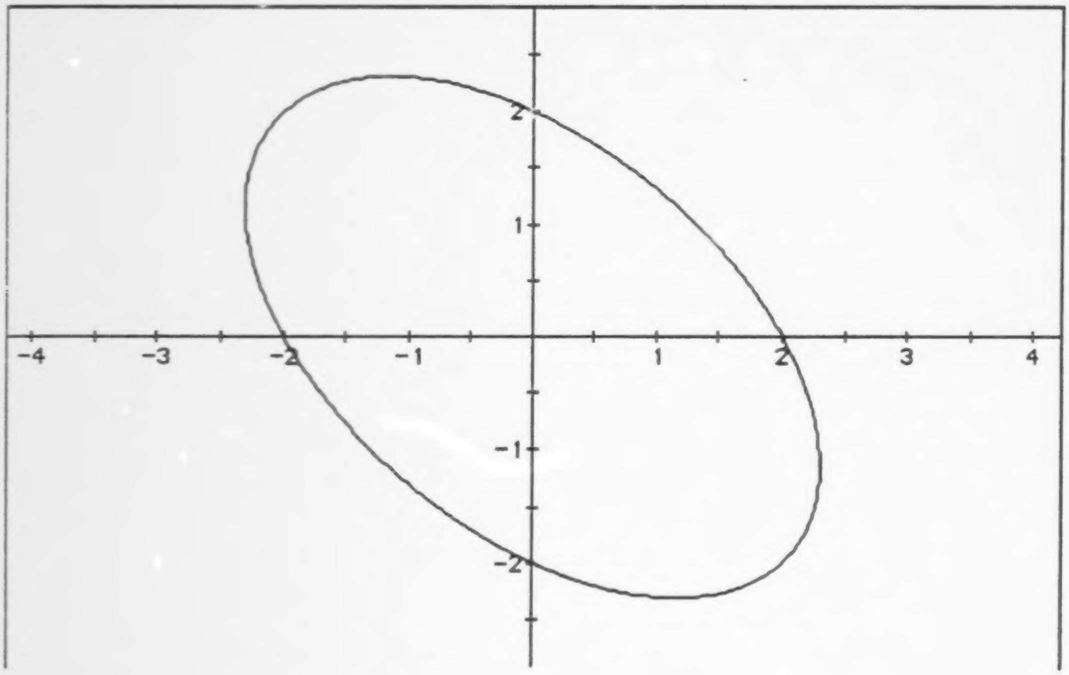


Figure 1. Graph of $y^2 + xy + x^2 = 4$

Next, consider any line with a graph passing through this ellipse. A simple case would be $y = x$. But, rather than graphing the ellipse and then the line as an overlay, consider the following relation

$$(y - x)(y^2 + xy + x^2) = 4(y - x)$$

The graph of this relation will be the ellipse and the line through it. Figure 2 is a graph of the new relation.

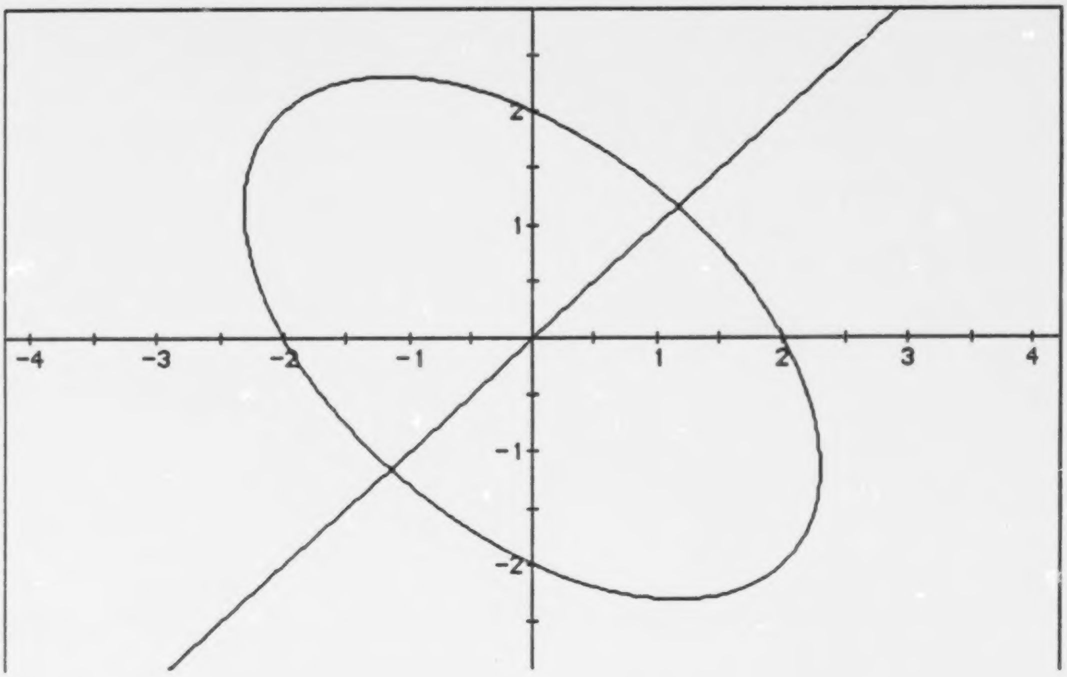


Figure 2. Graph of $(y - x)(y^2 + xy + x^2) = 4(y - x)$

Using some algebra, this latter relation is transformed into

$$y^3 - x^3 = 4y - 4x.$$

A family of curves can be generated by considering

$$y^3 - x^3 = ay - bx$$

where a and b are real numbers. In fact, if $a = b$ and both are positive, then all that changes from Figure 2 to a new graph of $y^3 - x^3 = ay - bx$ is scaling. For example, the three graphs

$$y^3 - x^3 = 4y - 4x$$
$$y^3 - x^3 = y - x$$
$$y^3 - x^3 = .5y - .5x$$

are shown in Figure 3.

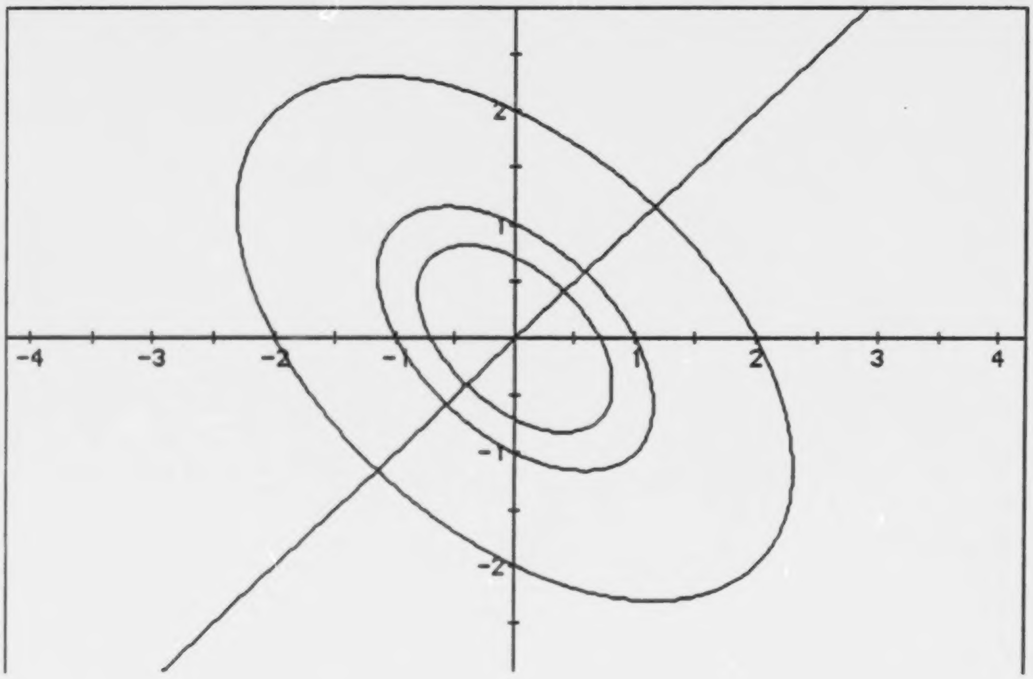


Figure 3. Graph of $y^3 - x^3 = ay - bx$ where a = b = 4, 1, and .5

Much more interesting, however, are the curves where $a \neq b$. For example the curve for $y^3 - x^3 = 3y - 4x$ is in Figure 4.

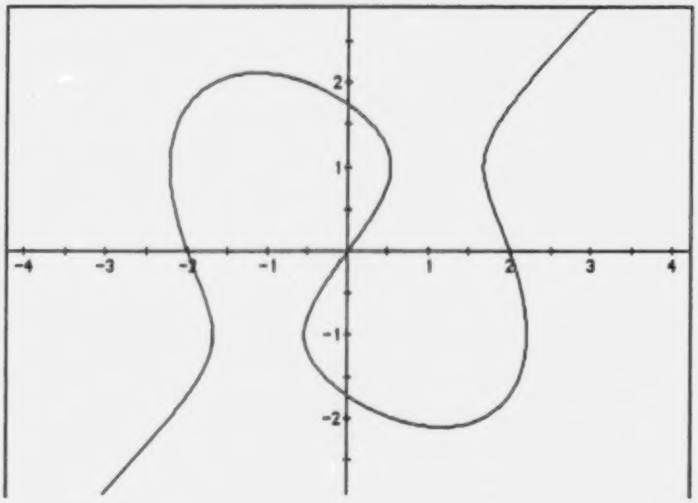


Figure 4. Graph of $y^3 - x^3 = 3y - 4x$

In Figure 5, a = -4 and b = 4.

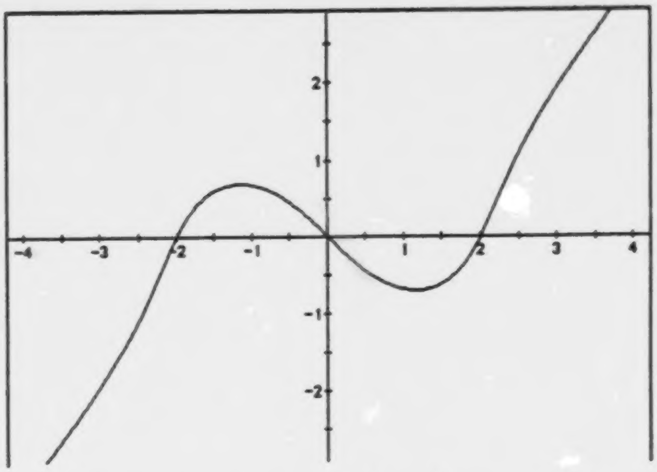


Figure 5. Graph of $y^3 - x^3 = -4y - 4x$

In Figure 6, a = 0 and b = 4.

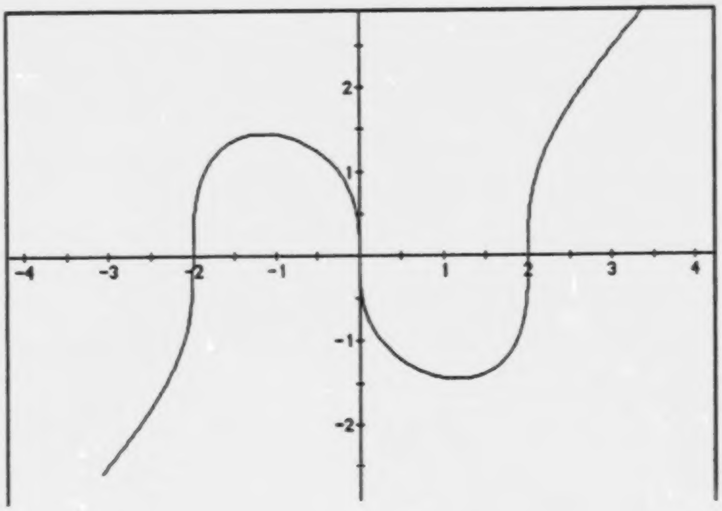


Figure 6. Graph of $y^3 - x^3 = 0y - 4x$

In Figure 7, graphs for a = .5, 1, 2, and 3 are given for b = 4 in each case.

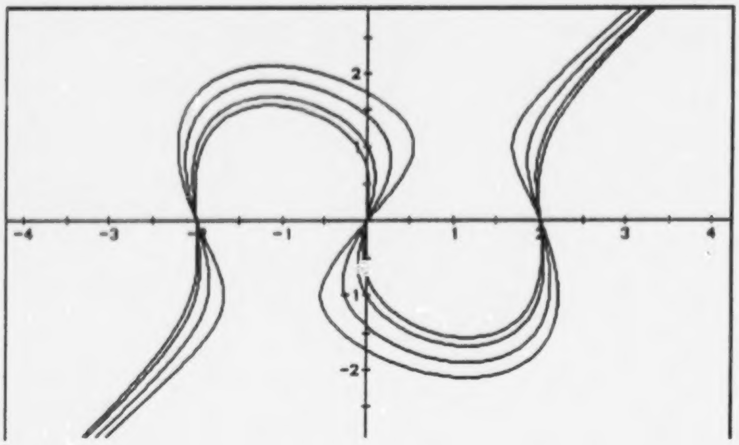


Figure 7. Graphs for a = .5, 1, 2, 3; b = 4

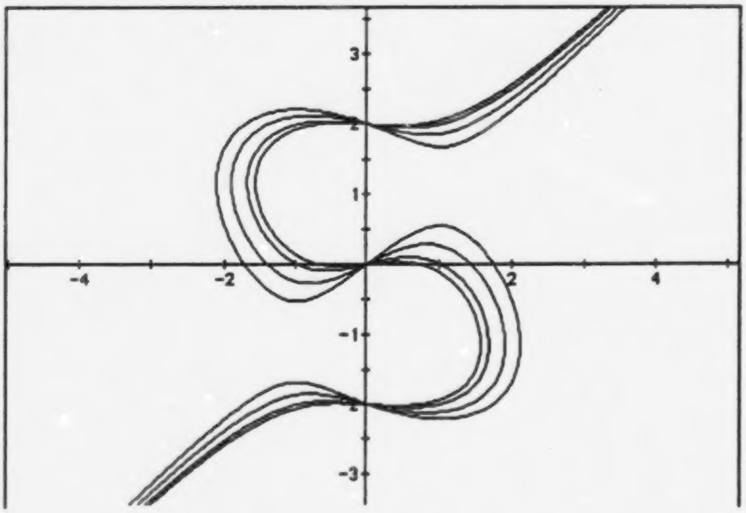


Figure 8. Graphs for b = .5, 1, 2, 3; a = 4

Perhaps the interesting cases are where a and b are nearly equal. Figure 9 is for $a = 3.9$ and Figure 10 is for $a = 4.1$, with $b = 4$ in each case.

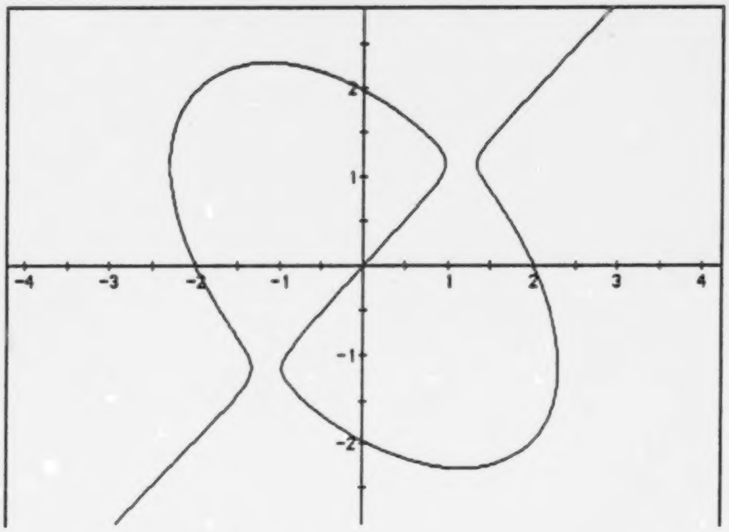


Figure 9. Graph of $y^3 - x^3 = 3.9y - 4x$

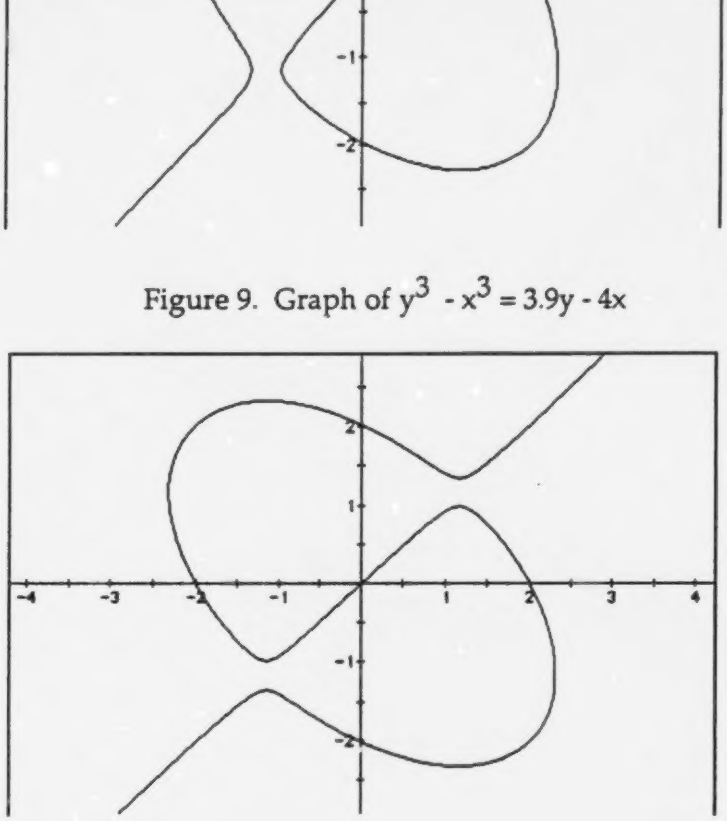


Figure 10. Graph of $y^3 - x^3 = 4.1y - 4x$

Figures 11 and 12 each show a set of curves that can be produced by varying different coefficients. Each is a set of 21 graphs where a coefficient has been stepped over 2 units in steps of 0.1. Figure 11 varies the coefficient of the x^3 term from 0 to 2; Figure 12 varies the coefficient of x from 3 to 5.

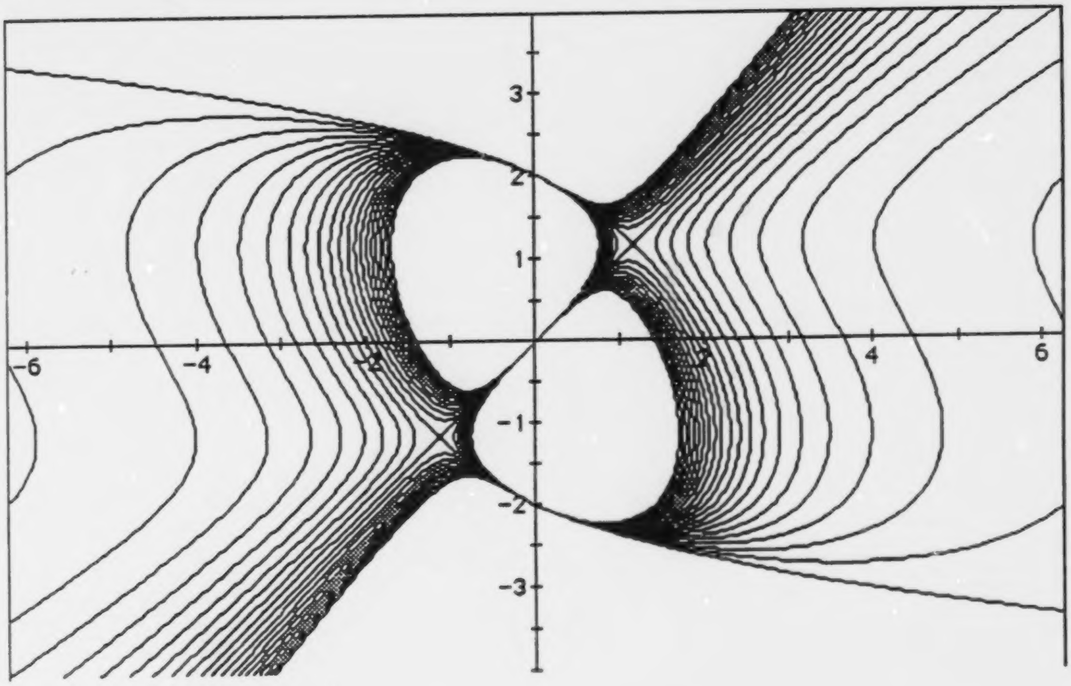


Figure 11. Varying the coefficient of x^3

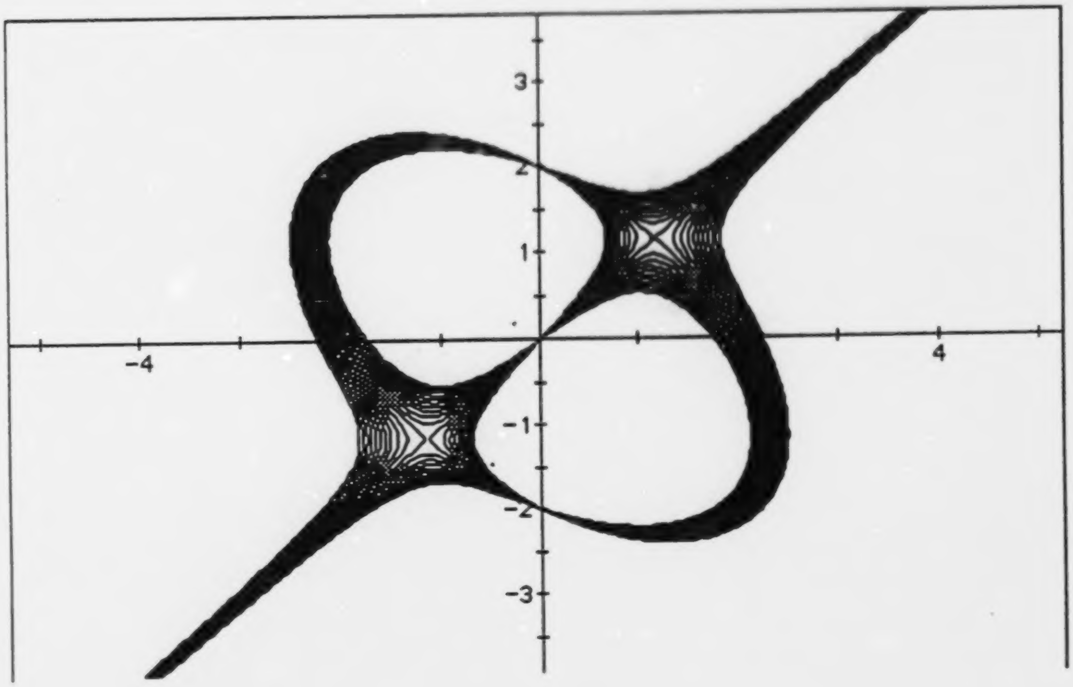


Figure 12. Varying the coefficient of x

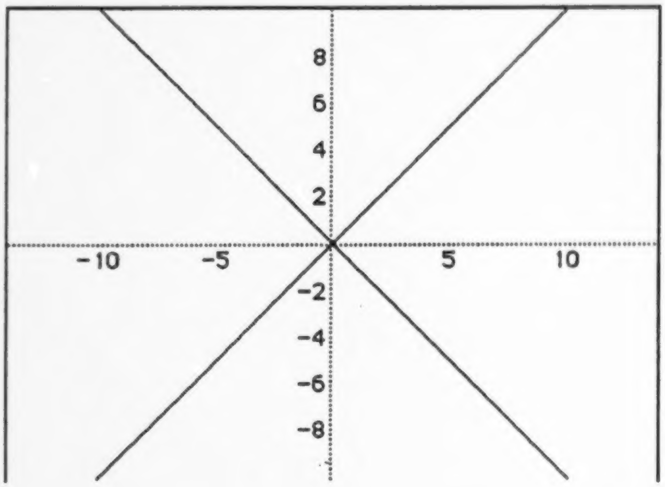
There is more to the investigation of these curves stimulated by the look at $(y^2 - 1)y = (x^2 - 4)x$, but let us turn our attention to applying the technique of multiplying relations to build composite curves that can in turn be modified to build a family of curves. Consider a very simple case, beginning with the equations for two lines:

$$y - x = 0$$
$$y + x = 0.$$

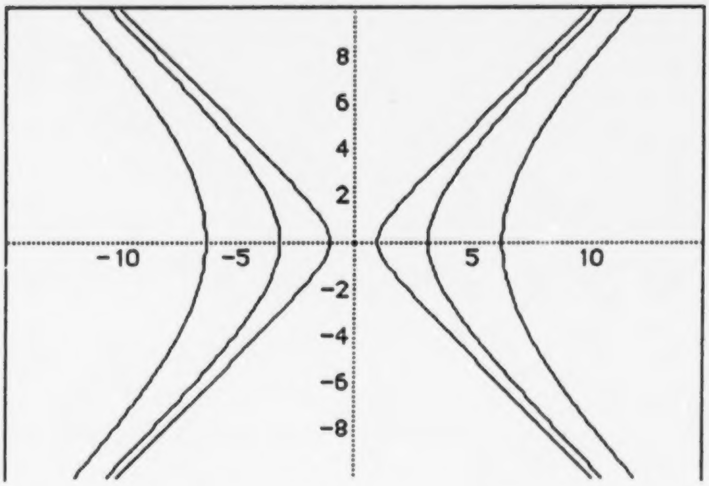
The product of the two relations is

$$(y - x)(y + x) = 0$$

and the graph is

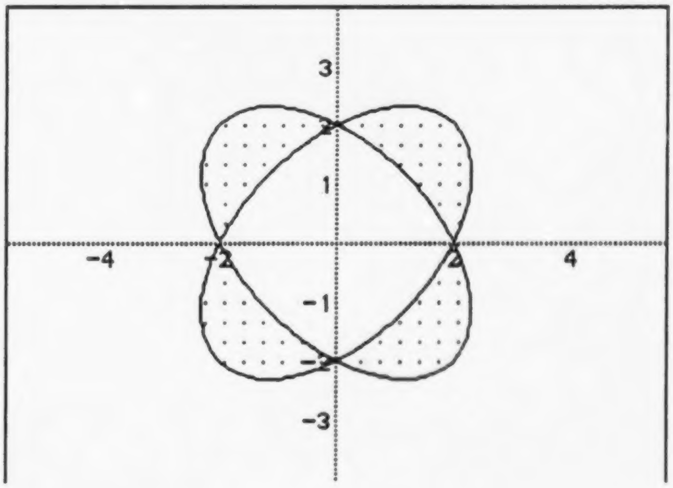


If the product is set equal to some constant other than 0, the graph is a hyperbola from the family of hyperbolas having these two lines as asymptotes. The following graph shows the graphs for three hyperbolas in this family. Similarly, if any other lines were selected, the technique could generate a family of hyperbolas.



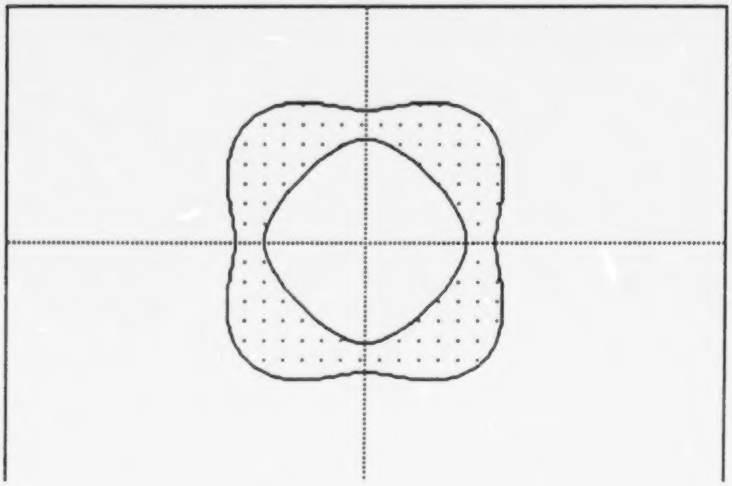
$$(y - x)(y + x) = -1$$
$$(y - x)(y + x) = -10$$
$$(y - x)(y + x) = -40$$

The technique can be used to build families of curves for which new conjectures and problems can be formed. For example, if the component curves are two overlapping ellipses, then the entire family of curves may be bounded. It is illustrative, but not necessary, to examine inequalities rather than equations. Let the product be

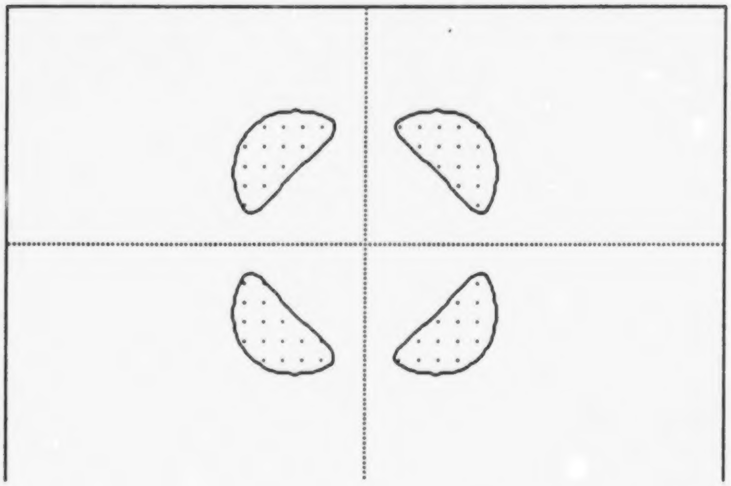


$$x^4 + x^2y^2 - 8x^2 + y^4 - 8y^2 + 16 \le 0$$

If the constant term is replaced by 15 or by 17, the following graphs are found.



$$x^4 + x^2y^2 - 8x^2 + y^4 - 8y^2 + 15 \le 0$$



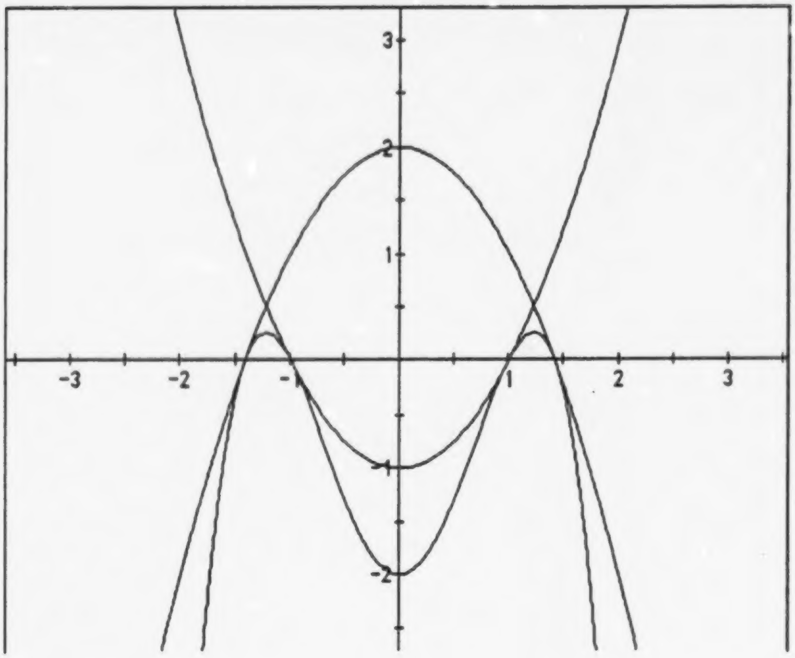
$$x^4 + x^2y^2 - 8x^2 + y^4 - 8y^2 + 17 \le 0$$

As the constant term is replaced by numbers lower than 15 the region in the center gets smaller and the shape approaches a square with the corners rounded. When will the center region reduce to a point? Why?

As the constant term is replaced by numbers larger than 17 the four small regions get smaller. When will they reduce to four points? What are the points? Why?

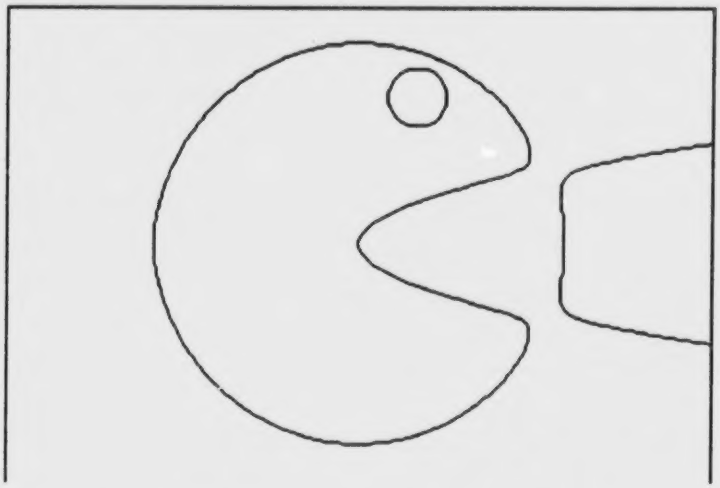
Consider an extension. Which of the two approaches, from Scenario 1 or from Scenario 2, might be extended most easily to find h(x) = f(x).g(x) such that f(x) and g(x) are each doubly (i.e., at two points) tangent to h(x)?

A graph:

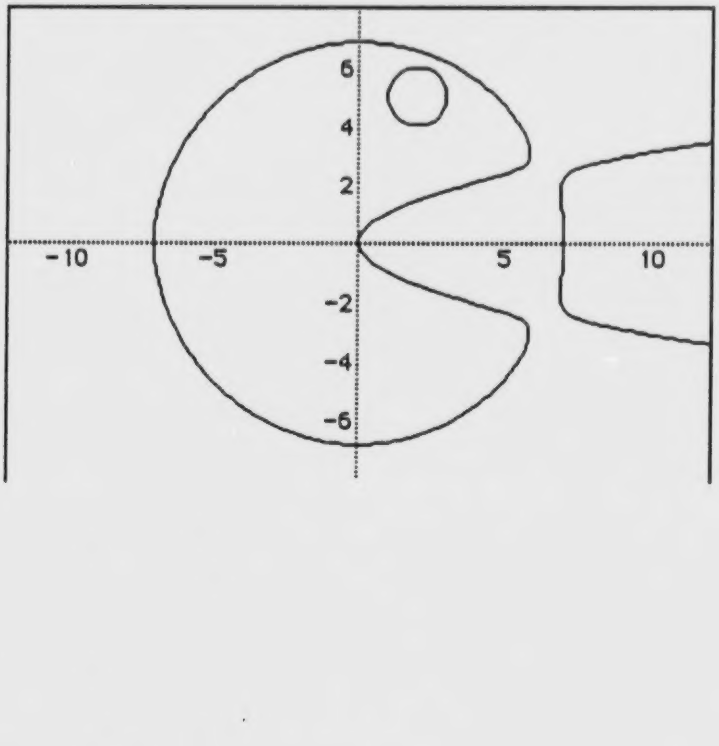


Finally, turn the page for one more problem . . .

What is the equation for the following graph?



Does this help?



An Investigation using Parametric Equations

James W. Wilson

PROBLEM:

Find the locus of the third vertex of an equilateral triangle when two of its vertices are moved along the x-axis and y-axis respectively.

INVESTIGATION:

Try this first by cutting out a triangle and physically rotating it along the axes, marking the locus of the third vertex. Can this physical movement be animated on the computer?

Repeat with other triangles -- scalene, isosceles, obtuse, acute, right . . .

Note that a scalene triangle could generate 6 different loci, depending on which vertices are along the axes and the orientation of the triangle. How are the six related?

Special case: The locus of the 90 degree angle in a right triangle when the vertices of the acute angles are on the axes.

Can you prove the locus is an ellipse (or a degenerate ellipse)?

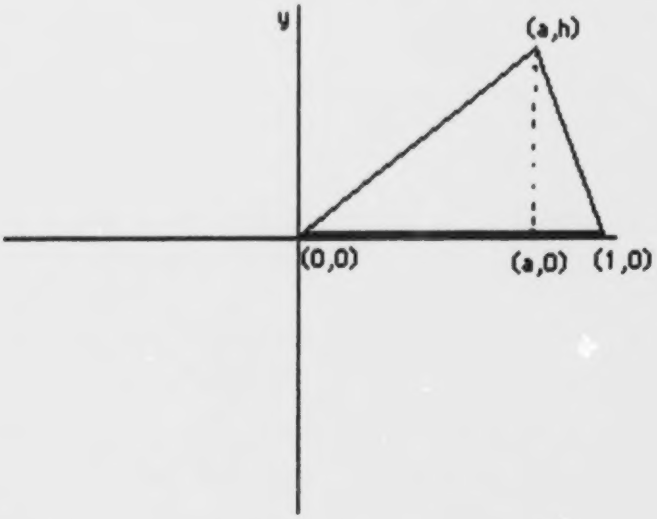
Consider the "special case" of a triangle with zero height. In other words, what is the locus of a point on a fixed line segment as the segment is rotated with its ends on the x and y axes? (Simplest cases: the loci of the end points. Next simplest case: the midpoint.)

OBSERVATION:

A mechanical device that physically rotates in this way is cutter for oval openings in picture matting. There is also an adult toy, sometimes called "the vacuum grinder," that uses the same principle. I have one on my desk.

ANALYSIS:

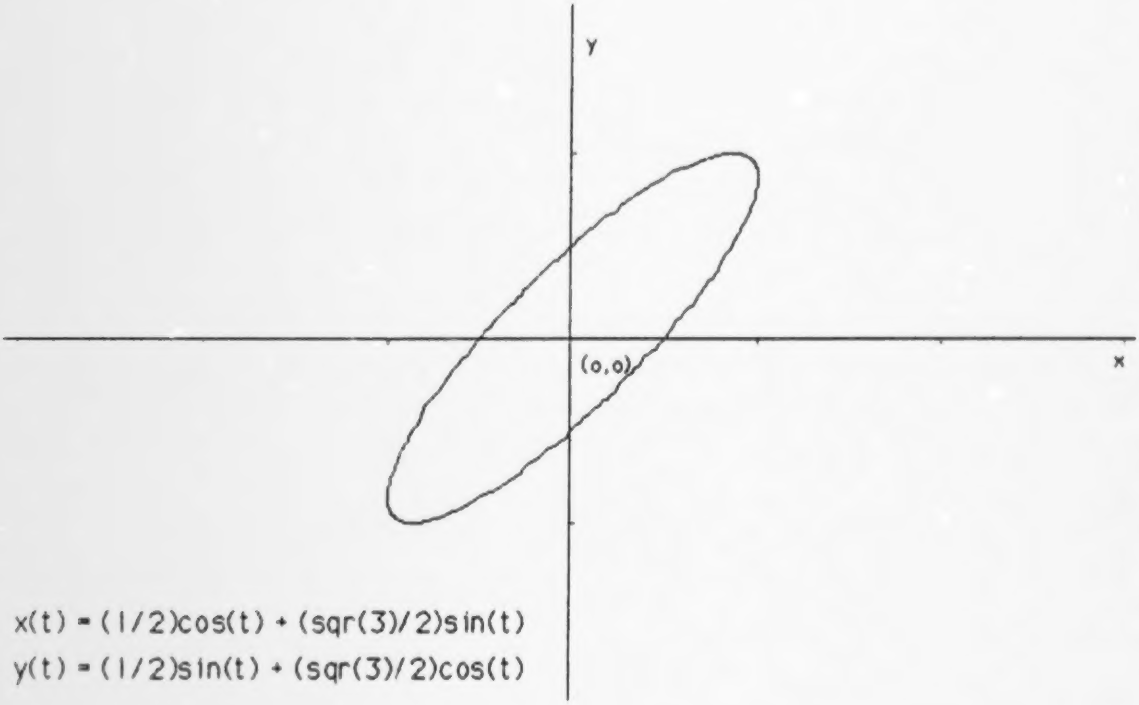
Let the base of the triangle be positioned with vertices at (0,0) and (1,0) for the initial position of the triangle. Let the third vertex be at (a,h). Therefore, the altitude of the triangle is h and the projection of the vertex onto the x-axis is a.



As the triangle rotates, let t be the angle the base makes with the x-axis. Then parametric equations for x(t) and y(t) are: (proof left as an exercise for the reader!)

$$x(t) = a\cos(t) + h\sin(t)$$
$$y(t) = (1-a)\sin(t) + h\cos(t)$$

For the case of the equilateral triangle, we have $a = 1/2$ and $h = sqr(3)/2$. The graph is



$x(t) = (1/2)\cos(t) + (sqr(3)/2)\sin(t)$

$y(t) = (1/2)\sin(t) + (sqr(3)/2)\cos(t)$

Please continue the INVESTIGATION using a function grapher.

What if $a = 3/4$, $h = 2$?
$a = 5/4$, $h = 1$?
$a = -1/4$, $h = 1$?
$a = 1/2$, $h = 1/2$?

What if the vertices for the base of the triangle rotated along two lines intersecting at a 60° angle?

Note: The above graph was generated using Mathematics Teacher Workstation from Sterling Swift Software, Austin, Texas. The author is Ray Carry.

Discussion of Professor Wilson's Paper:

Nohda: Thank you very much, Professor Wilson. You have given a lot of demonstrations. However, we now have very little time remaining for discussion, so please ask your questions and give comments. Professor Becker.

Becker: Jim, how difficult would it be for you to bring up the rectangle problem again, where you had the three triangles with equal areas?

Becker: Thank you. Now, I think that's a very nice demonstration. I think it could be used in the elementary school curriculum or in inservice teacher education; what was the situation in which you use or have used the software?

Wilson: This was a class of approximately forty pre-service elementary teachers. They are probably mostly sophomores. They would have had only one other university level math course. Many of them had had more, but they were required to have only one other university level math course and that would have been the course on development of number systems. So this was for many of them this course was their first exposure to geometry since high school. This problem I brought in after we had been doing some things with geoboards and after we had been talking about area and, in fact, after we had looked at the proposition that area of a triangle is altitude times base times 1/2 and as the vertex of the triangle is moved, the area stays the same. We had not looked at it on the screen. This is the first context in which we looked at it on the screen.

Becker: So what they're seeing is that the base and the height are invariant.

Wilson: That's right. That's what I would hope they see.

Becker: One of the interests I have in further U.S.-Japan collaboration is looking at software like this and then determining exactly how it would best be used in the elementary school curriculum; that is, how does it fit in with the content that is taught? I have two other comments. The student can think about this problem you've demonstrated in several different ways which I think is quite nice. Finally, would we regard the movement of the three triangles to show that they occupy half the area of the rectangle as a proof?

Sawada: First of all, the attitude towards problem solving is very inspiring and encouraging. Now, coming back to the software demonstration, when you introduce this in high school geometry, how can you use it, because in many cases the teachers feel that they must be able to prove whatever propositions that arise from the students and this may be hard for the teachers. So they may have this fear and it may be difficult to use the software.

Wilson: There are several things in that statement. First of all, I'm using a generic kind of general piece of software, the Geometer's Sketchpad. As the teacher I have a lot of input into putting together this kind of exercise, so it is mine. But the Sketchpad is a tool. It is not specifically a piece of software that does this. It's a general software. Second, the comment about proof - certainly I want demonstration and I want proof. I think this kind of exploration always cries out for following up. If we come to a problem that I cannot do the proof on, that doesn't mean it doesn't get put before the class. And I want my teachers to have the comfort with saying to them "I don't know how to prove either. Let's work on it." I think we have to get beyond that point of saying if I don't know the answer then we can't use it in class.

Damarin: Jim, it seems to me that you were speaking about demonstrations. This is teacher-led problem solving, is that primarily what you were saying?

Wilson: Well, certainly teacher presentation, but I think I always have things that I want students to do that follows up on it.

Damarin: Then my question, really following on that, is what resources do students get with this so that they can perhaps review a lesson at home or that kind of thing?

Wilson: For this particular problem, we had a Macintosh lab available to us and one of my graduate students was available if they wanted someone to talk to about it, plus I had handout materials that went with it. Now many of the things that we do explorations with, like Theorist, explorations with other software over at the high school, we will have some sort of guidelines or materials for students to look for. And we tailor that to the particular lesson and to the particular problem, but there's usually some sort of materials or handout that is available and the software is on the computers in the laboratory.

Nohda: Well, I think there is no time left. Thank you very much.

End of Discussion

MATHEMATICS EDUCATION IN A HIGH-TECHNOLOGICAL INFORMATION-ORIENTED SOCIETY--WHAT SHOULD IT BE?

A summary of the Report by The Executive Committee of JSME in 1987 and some comments by the chairman

Tsuneo Uetake
Asia University
Tokyo, Japan

0. Introduction

School mathematics should be re-examined from the standpoint of "What is the mathematical literacy that the average citizen must have in a high technological, information-oriented society?" I will consider about three important themes (1) - (3):

(1) What sort of ability must be kept in the face of a changing society?

(2) What sort of ability must be developed for a high-technological information oriented society?

(3) How far should the average citizen study the contents of black-boxes, which are increasing with the development of computer and means of communication?

1. Behavioral objectives in mathematical education

I will classify the behavioral objectives in mathematical education into four "ability" levels ranging from the least difficult to the most difficult, according to IEA (International Association for the Evaluation of Educational Achievement):

COMPUTATION:	Execution by routine procedure
COMPREHENSION:	Execution upon the understanding of meaning or/and principle.
APPLICATION:	Problem-solving by routine procedure using suitable knowledge or operation.
ANALYSIS:	Solving problems that cannot always be solved through routine procedure.

1-1 Many of the objectives at the "computation" level will become meaningless with the development of the computer, because the computer is far superior to human beings in the speed, the exactness and the capacity at which it executes these objectives. So, it is desirable to select carefully fundamental objectives among them and raise them to the "comprehension" level.

1-2 At the "computation" level, the following objectives should be retained:

Mental addition, subtraction, multiplication between 1-digit numbers
Addition, subtraction, multiplication between 2-digit integers on paper.
Drawing of basic figures on paper with simple tools.

1-3 At the "comprehension" level, it is important that the ability to "visualize" be retained: Connect an object of study to some figure or scheme corresponding to it.

Examples: A rectangle in multiplication of rationale;
A number line in addition or subtraction between signed numbers.

1-4 From a standpoint of humanistic education, it is important not only to the intellectual side of mathematical literacy, but also to the affective side of it.

2. About the "Black-box"

2-1 If we re-examine our objectives from the standpoint of the learning about black-box, there are various approaches, and these will be useful for mathematical education in a new society.

2-2 The algorithms in the Black-box can easily be understood if they contain only simple computation : it will be helpful as a basis for information science, and using the computer will be effective for the learning.

- Some examples of "transparent" box in traditional school mathematics:
  Linear, quadratic, and rational functions,
  Operations of numbers, vectors, matrices and formulas,
  Linear transformations,
- Examples of the black-box in traditional school mathematics:
  Square root and trigonometric, exponential and logarithmic functions (as tables of these functions)

2-3 Almost all operations (computations) that are done with "paper & pencil" in traditional mathematics are entering into black-boxes as computer software. So, for "complicated computations", we must positively use the computer as a tool in the form of black-box instead of these computations. It is necessary to develop the black-box (software) suitable for a tool in mathematics education.

Examples: Electronic spreadsheets (see Appendix)
Formula manipulation systems
Computer graphic systems

2-4 When we use a black-box, it is indispensable to learn about its meaning.
The traditional behavioral objectives at the computation level may be re-evaluated as a behavioric experience for the understanding of its meaning. However, we should limit such experience to the elementary and/or typical level. Therefore, it is not necessary to become proficient in these manipulations.

2-5 In using a black-box as a tool, it will become important to achieve the ability to understand the meaning of the input or output of it, in order to achieve one aspect of mathematical literacy.

3. Problem solving and logical thinking

3-1 I think that traditional Japanese mathematical education has mostly come to halt at the "Application" level. It is now desirable to raise it to the "Analysis" level.

3-2 In order to develop problem-solving ability, it is desirable to prepare as many good problems as possible that are suitable for each school level and that are mathematically meaningful. Problems of this kind have already been developed both in Japan and in foreign countries, but hereafter we should give priority to accumulating typical problems suitable for use with the computer as a tool.

3-3 In a high-technological information society, it is important to develop students' ability to solve problems and think logically, but these educational objectives at a high level are too difficult for most students.

3-4 Some people say that computer programming is effective for developing the ability to think logically; however, this hypothesis does not yet have sufficient conclusive evidence to support it.

3-5 The reasoning in Euclidian geometry is deeply connected to mathematical intuition through the figure, so it is very dangerous if students lose the chance to come into contact with it.

4. Use of the computer

4-1 By using the computer as a tool, it may be possible for us to deal with teaching in a more liberal way, apart from the traditional framework.
[I am referring to the situation as it exists in Japan] So, it is desirable to accumulate typical problems of this kind.

4-2 If there comes an age in Japan when we can freely use computers inside and outside the schools, a prescribed program of school mathematics may become meaningless.

4-3 For the time being, the important theme in using computers in school mathematics is how we can use them effectively as a tool for problem solving in the general classroom.

For example: Using a large screen or digital display for demonstration
Group learning via several machines
Using notebook size computers

4-4 Almost all CAI (Computer Assisted Instruction), stand-alone or network, as individual learning, is used for "drill and practice".

4-5 At the present time, almost all authoring systems are unsatisfactory for mathematics education. For effective use of these systems, it is necessary to accumulate the classroom experiences mentioned in 4-3.

4-6 CMI (Computer Managed Instruction) for educational information should be used as a technique for educational research rather than a way to save time.

4-7 Education for computer literacy should not only be included in the teaching of programming languages but also be made a part of the attitude and ability that recognize the computer as a tool and which encourage the appropriate use of that tool in every field. Mathematics education can share in such education.

Appendix 1

NEW CURRICULUM (FROM 1994) FOR UPPER SECONDARY SCHOOLS IN JAPAN

(Mainstream)	(Side options)
**MATHEMATICS I(4)	*MATHEMATICS A (2/4)
Quadratic functions	Numbers and formulas
Trigonometric ratios	Plane geometry
Enumeration	Number sequences
Probability	Computation and Computer
*MATHEMATICS II(3)	*MATHEMATICS B(2/4)
Various functions	Vectors
Figures and equations	Complex numbers and Gauss plane
Introduction to analysis	Probability distributions
	Algorithms and computers
*MATHEMATICS III(3)	*MATHEMATICS C (2/4)
Function and their limits	Matrices and linear computations
Differentiation	Various Curves
Integration	Numerical computation
	Statistical Processing

Note: ** Required subjects; * Optional subjects; ( ) Number of credits

Appendix 2

TEACHING OF LINEAR ALGEBRA USING AN ELECTRONIC SPREADSHEET

1. Introduction

All the content in the "Mathematics B" are learned using the computer as a tool in the form of black-box. This paper shows how "Multiplan" can be used in the teaching of the content of "Matrices and linear computations" in this subject.

Matrices and linear computations

- (a) Matrices
  - a. Matrices and their operations
    addition, subtraction, multiplication by a real number
  - b. Product of matrices and inverse of 2x2 matrices
- (b) Simultaneous linear equations
  - a. Representation by a matrix
  - b. Solving by sweep-out method

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2. Matrices and their operations

At the definition of matrix, we can use the row and column on Multiplan, and "programming" for addition, subtraction, multiplication by a real number are very easy if students use the relative representation of variables and the "copy" function on spreadsheet. For the following example, all elements of A + B, A – B are RC[–8]+RC[–4], RC[–12]–RC[–8], respectively.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1		A				B				A+B				A–B	
2	2	1	2		1	1	3		X	X	X		X	X	X
3	3	2	4		0	1	2		X	X	X		X	X	X
4	5	4	3		3	2	1		X	X	X		X	X	X

3. Product of matrices

If 2 matrices A, B are given the following, (1,1),(1,2) elements of A B are

RC[-3]*R[-3]C+RC[-2]*R[-2]C+RC[-1]*R[-1]C

RC[-4]*R[-3]C+RC[-3]*R[-3]C+RC[-2]*R[-1]C,

respectively.

These formulas will be able to "write" easily by the moving of the cursor on spreadsheet.

1	2	3	4	5	6
				B	
			1	1	3
			0	1	2
	A		3	2	1
2	1	2	X	X	X
3	2	4	X	X	X
5	4	3	X	X	X

4. Simultaneous linear equations

a. Representation by a matrix

A matrix

$$\begin{matrix} a_1 & b_1 & c_1 & d_1 \\ a_2 & b_2 & c_2 & d_2 \\ a_3 & b_3 & c_3 & d_3 \end{matrix}$$

represents the next simultaneous linear equation

$$a_1\ x_1 + b_1\ x_2 + c_1\ x_3 = d_1$$

$$a_2\ x_1 + b_2\ x_2 + c_2\ x_3 = d_2$$

$$a_3\ x_1 + b_3\ x_2 + c_3\ x_3 = d_3$$

b. Solving by sweep-out method

This "program" can use the"copy" function on all rows. I will show an example of the"sweep-out program" on Multiplan to solve the above equation.

```
"ﾚﾝﾘﾂﾎｳﾃｲｼｷ"
2                    1                    1                     2
2                    -3                   -5                    8
1                    -2                   1                     9

R[-4]C/R2C1          R[-4]C/R2C1          R[-4]C/R2C1           R[-4]C/R2C1
R[-4]C-R[-5]C*R3C1/  R[-4]C-R[-5]C*R3C1/  R[-4]C-R[-5]C*R3C1/   R[-4]C-R[-5]C*R3C1/
R2C1                 R2C1                 R2C1                  R2C1
R[-4]C-R[-6]C*R4C1/  R[-4]C-R[-6]C*R4C1/  R[-4]C-R[-6]C*R4C1/   R[-4]C-R[-6]C*R4C1/
R2C1                 R2C1                 R2C1                  R2C1
R[-4]C-R[-3]C*R6C2/  R[-4]C-R[-3]C*R6C2/  R[-4]C-R[-3]C*R6C2/   R[-4]C-R[-3]C*R6C2/
R7C2                 R7C2                 R7C2                  R7C2
R[-4]C/R7C2          R[-4]C/R7C2          R[-4]C/R7C2           R[-4]C/R7C2
R[-4]C-R[-5]C*R8C2/  R[-4]C-R[-5]C*R8C2/  R[-4]C-R[-5]C*R8C2/   R[-4]C-R[-5]C*R8C2/
R7C2                 R7C2                 R7C2                  R7C2

R[-4]C-R[-2]C*R10C3  R[-4]C-R[-2]C*R10C3  R[-4]C-R[-2]C*R10C3   R[-4]C-R[-2]C*R10C3
/R12C3               /R12C3               /R12C3                /R12C3
R[-4]C-R[-3]C*R11C3  R[-4]C-R[-3]C*R11C3  R[-4]C-R[-3]C*R11C3   R[-4]C-R[-3]C*R11C3
/R12C3               /R12C3               /R12C3                /R12C3
R[-4]C/R12C3         R[-4]C/R12C3         R[-4]C/R12C3          R[-4]C/R12C3
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Appendix 3

CONTENTS OF DISCRETE MATHEMATICS AS THE BASIS OF INFORMATION SCIENCE

(From the proposal of the report by the executive committee of JSME)
*Enumeration, *Positional notations (Binary & Hexadecimal numbers), *The idea of Algorithm (esp. Recursive programs), Number theory, Graph theory.
*is adopted partially into the new Japanese curriculum.

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Discussion of Professor Uetake's paper:

Teague: Thank you, Professor Uetake, for a very interesting and inspiring talk. The floor is now open for questions/comments. Professor Becker.

Becker: Thank you. On page 4 of Professor Uetake's paper in paragraph 3.4, he says "Some people say that computer programming is effective for developing the ability to think logically. However, this hypothesis does not yet have sufficient conclusive evidence to support it." I'm wondering about the extent to which in Japanese schools, now or in the near future, students actually write programs for the computer to solve particular problems. How widespread is this, and is it considered important?

Uetake: In Japanese high schools, mainly the BASIC programming language is introduced. While the meaning of what logical thinking is not clear and is a problem, it depends on how you interpret it. But, in this case the logical thinking is the meaning given to logical thinking in item 3.5.

Choate: I have a question about 3.5. You say the reasoning in Euclidean geometry is deeply connected to mathematical intuition through the figure. Could you explain to me what you mean by "through the figure?"

Uetake: Well, I feel that there seems to be a difference in determining what "proof" and "reasoning" mean. In Japan, unless a student can actually write down the whole proof, we cannot say that the student has got the reasoning for it. So we mean that the student must have the reasoning up to the level that you really are able to write down the whole proof, supported by geometrical intuition.

Choate: Following up on that, in America we seem to be obsessed with proof in geometry to the stage where students have to write proofs that contain all axioms and all reasons have to be very carefully presented. Is this what is done in the Japanese schools?

Uetake: Well, in Japan too, when it comes to proof you have to write down everything. But I feel that after looking at and observing all the demonstrations so far, what is given seems like reasoning used only as just introductory to proving something. It's not the final stage.

Ferrio: Professor Uetake, in the United States for some time now we have had a lot of training in computer programming in traditional languages like BASIC and Pascal; but now with new software programs like we've seen this morning, like Excel, tools like that that are very powerful and have script capabilities and things like that. I think my opinion is that the emphasis in the United States is changing to teaching students how to use those powerful tools which have the programming built in and there's less emphasis on the traditional, formal programming languages that computer scientists use. I have two questions related to that. One is what is your opinion on that? Number two is about the new Japanese curriculum, will it continue to emphasize traditional programming or will it be adaptable to some of these new tools?

Uetake: First of all, when the computer was originally introduced, most of the time the BASIC language was attached to it. And that's why we started using BASIC because buying new software is expensive. But certainly the more the software is available, well, I think we should maybe go in that direction. And for the second question about curriculum, in Math A and Math B in item 4, emphasis is placed on programming itself, whereas in Math C the use of commercial software is encouraged.

Dugdale: On page 5 item 4.6, you mention the use of computer-managed instruction as a technique for educational research. Could you say more about your ideas of how that would work, of how you would use computer-managed instruction for research?

Uetake: What we call the response analyzer, the CMI software, it is available and the teachers are using it. And they get all the data of the students' responses and then they analyze the students' activities. They are using this to save time. I am saying that that is not a good way of using CMI.

Kaida: I have a question on page 4, #4.2. It mentions that prescribed programs of school mathematics may become meaningless, and I'm wondering what does that mean...the prescribed program?

Uetake: The program doesn't mean the computer program. It means the curriculum.

Kaida: So it means the mathematics curriculum itself. I have a follow-up question then. I think for many of our students in the United States, at least in our school, I think we could say that they do have free use of computers at home and at school, for a majority of them, and yet we have a prescribed program that we still follow. So how do you view that that would be different in Japan from the United States?

Uetake: In Japan it may be worse and if this situation continues, then it would be really bad. So something should be changed in the future. That's the reason I wrote this paper. Of course, there are a lot of very active teachers who are trying to introduce a new way of teaching and some of them have been introduced here.

Fey: I think this point highlights a difference that's been running throughout the presentations of the conference and it's a difference between Jim Wilson's talk this morning and some of the other, some of the typical curriculum talk in our country. It seems to me that what Jim meant by solving problems or problem solving was exploring mathematically rich situations and finding interesting ideas there. And what is sometimes meant by problem solving is more at the application level in your discussion, taking a technique that you know, a mathematical technique, and applying it to a particular well defined problem. I sense that what is made possible by the technology is to give students access to much more of the exploratory style of learning because there isn't the hierarchy of prerequisite mathematical skills that have to be mastered before you can investigate the situations. In that sense, what Professor Uetake is talking about is made possible by the technology, but it raises a very fundamental question for us in thinking about what is the curriculum? What are our goals in the curriculum? Is it a well defined set of specific content objectives? Or is it an environment in which students explore and develop abilities that are much less sharply defined and limited?

Uetake: In the curriculum given by the Ministry of Education, the general objectives for math education are stated. And they include both application and the idea of creativity. But whether those goals are achieved or not is a different question. The problem is that, in Japan, math education is greatly influenced by the entrance examinations, as you've heard. Recently an article was printed in the Shukan Asashi, a weekly magazine, July 5, 1991, and it asked whether math is memorization or a creation of mind. And a response was given by Peter Frankl who is an International Mathematics Olympiad winner. He and others were given the math problems and he presented a very unique way of solving a problem. And so, I think it is of great interest. But at the same time, for about ten years now, a book called Mathematics Is

Memorization is being sold and is very popular; so, there is a problem.

J. Wilson: I know many of our Japanese colleagues are members of NCTM and if the election had gone the other way I'd be campaigning about the Standards and presenting those; but I would say that much of the spirit of what I see this curriculum under construction with these teachers and students constructing their mathematics and following up on interesting ideas is underlying (undermining?) the philosophy that's in the Standards. Now, there is no question that what goes on in our schools is not described in the Standards. What is intended there is that that should be out, and what we should work towards over the next decade or more. And I would hope that you examine the Standards for Curriculum and Evaluation and the Standards For Teaching and comment to us in the United States about those and see how over time those things play out both here and in your way of thinking too.

Uetake: Most Japanese scholars, mathematics educators and mathematicians are very studious and I'm sure they have already read NCTM's Standards.

Wilson: I wish I could say the same for the U.S.

Uetake: But, I am specifying only the researchers and scholars. I don't know anything about the math teacher in the classroom, in this respect.

Miwa: Probably we will discuss the curriculum this afternoon, but in Japan the standard curriculum is set by the Ministry of Education which includes the goals and objectives, as Professor Uetake has said, so certainly we have definite objectives and specify the content. Every teacher must follow and cover all those objectives. There is no choice of selection. But as the use of computers gets more and more popular in the near future, this system may have to change. Also, in a way the emphasis on individualization is getting more and more popular and so this trend will go farther. The problem is that everything is based on Japanese culture which is not very liberal; so this movement may be quite slow, unfortunately. But, especially, in the math field it may be less likely: Another problem is that the curriculum is revised only every ten years, and certainly the curriculum should be revised more frequently in the future. This is the big problem in Japan right now.

Sawada: Probably you have noticed that, in Japan, math educators are mainly concerned with and working on the elementary and lower secondary school levels in mathematics. And we are succeeding in this area, whereas, it seems that in the States the main emphasis is on high school math education. In Japan, at the high school level, the main emphasis is still on acquiring information, in other words, memorization. This is due to the great influence of the entrance examinations. This should be changed and in Japan, also, we should have more emphasis on high school math too.

Damarin: Along these lines, I don't understand fully your high school curriculum, in particular, what is the relationship of mathematics II, mathematics III, and mathematics A, B, and C? Are there differences among the students who choose those different optional courses that you think about as you design the curriculum? And third, I guess as a follow up, how is that related to the differences in the role of the computer in those various curricula? It's sort of a big question, but can you respond?

Uetake: For one thing, by giving some options, the main purpose is to provide a variety of courses for students. Math I, II, III are the main courses and, especially, Math I is compulsory - everybody has to take it. Math II and III are for math and science majors. However, Math A, B, C are electives. Each course contains four items and a student can choose two out of four. That way they have a lot more options. The problem now is that when students take the college and university entrance examinations, they are at a loss of what to do. That's one of the reasons this system was developed; we want to stimulate the colleges and universities to do something about the entrance examinations. The Ministry of Education just very recently and suddenly announced that from next year on, every university and college can make their own curriculum; this is free and total liberation. This certainly is confusing to colleges as well as to the math teachers, and so when I go back to Japan I will work on that. But I have some anxiety because if this happens, then the college of liberal arts may disappear and I am fearful of that. So, you see, we have to really work hard on the new curriculum.

Teague: We are approaching our luncheon time, so I think it's time to bring this to a close. Thank you again, Professor Uetake, for a very stimulating presentation.

End of Discussion

SUMMARY OF SEMINAR

Miwa: Now we will begin the discussion concerning an overview of our Seminar and various other matters. We'll begin with Professor Sugiyama, who has some comments.

Sugiyama: To begin the discussion I'd like to say that we feel that we've learned a lot by looking at the work of our American colleagues. In addition to interesting and useful papers, we've seen quite a bit of very interesting software. We've learned a lot about using computer software for graphing, for example. We've seen a lot of software with a lot of imagination built into it, particularly yesterday we saw a lot of the kind of software that we really haven't seen much of in Japan, and we felt that it's full of explorative spirit. And at the very beginning of our discussion today, I'd like to thank the our American colleagues for all they've brought to the conference.

Damarin: Speaking for myself on the American side, we've learned a lot about sameness and differences between our two groups of mathematics educators. We've learned something about the meaning of problem solving in the two societies, something about curricular differences, and something about the ways in which maybe those differences help to shape the development of technology for curricular use. It seems that there are some major differences between the two societies in the constraints imposed by the curriculum, and maybe the conception of problem solving placed on the kind of software that we've developed to date and the ways that we think about implementing technology within our separate boundary conditions. To me those are really interesting areas to explore further. I'd be particularly interested in knowing more about how Japanese teachers take technology and use it within a more constrained kind of curriculum than we have, a more line-by-line spelled out curriculum than we tend to have. I was very interested in the papers of Professor Sawada and Professor Kumagai. We have in our country a lot of questionnaire studies of teachers and we've never, to my knowledge, asked teachers in all of those studies whether they talk to each other about technology. And I think we have a lot to learn from the ways you address teacher training, that Professor Hashimoto talked about, and I would be very interested in learning more about those. There's a whole lot to learn from each other and I appreciate being here very much.

Becker: Jon, are you interested in making some comments, since I know you have to leave for the airport shortly?

Choate: I leave here very humbled. I leave with far more questions than I arrived with. I'm intrigued by how technology could be so dependent on culture. I never really thought about it before. To be honest I go back challenging a lot of assumptions I've made about how people learn because I guess I never really asked the important question, which was what's global learning. Maybe that's a comment on my background. I leave very impressed with how our Japanese colleagues have taken technology and been able to shape it to their curriculum. I think we have a lot to learn from them. I wish I could take back with me some of your software and show it to some of my colleagues, particularly to those who think they have a lot of answers because I don't think they have the answers. Personally, I'm intrigued with the whole question of visualization and I've found many of the Japanese ideas about this intriguing. I can't help but believe that we're in an age where we learn with our eyes and I'm not sure we really know what that means. One final comment is I'm intrigued that I don't think that our Japanese colleagues have really started to address the whole question of what mathematics the technology brings with it. I wonder what happens in a system where the time delay is ten years because the most recent statistic that I heard is the amount of mathematical knowledge doubles every ten years. I find it interesting that there's no real mention of discrete mathematics and, in our curriculum, we have discrete mathematics now because that is the mathematics that came about when America really industrialized. In other words, it's the mathematics that our society has used to solve many logistics problems, many problems that are of the modern age. I always thought it necessary that we taught this to our children. I'm not so sure now, but I still think there may be something there. And the final thing I'd like to say and I ask my colleagues here this question, I wonder how many people have read the book called In the Age of the Smart Machine - the Future of Work and Power? It is written by a professor at the Harvard Business School and it is a study of what will the workers need to know in the year 2000. And what she comes up with is a picture of a worker sitting in front of a t.v. screen manipulating some images. Her concern is that the workers not treat the images as a black box, but that they be able to understand the image as being pictures of a dynamic process that they understand. I leave here still convinced this is what we need to know. I leave here realizing that perhaps there's a far different answer to it than I thought. I would like to thank all members of the Japanese delegation for a memorable learning experience.

Becker: The name of the book is In The Age of Smart Machines.

Hashimoto: As one member of the Japanese delegation I'd like to echo what Professor Sugiyama has said and thank everyone and mention further that we've learned quite a lot. In particular, there are two points that I think came up and should be emphasized. One thing is that in the classroom, when introducing computers and software, the teacher needs to have a spirit of fun and play when doing it. I think perhaps Professor Wilson's demonstration this morning would be a good example of that. Also, with respect to graphical software we've seen, what do we mean by graphical and isn't it possible that the emphasis on graphics will make it possible to create new school math; however, this may cause problems for the teachers, at least in the short run, but I'd like to think about the possibilities that this has brought.

Sugiyama: Earlier I expressed my appreciation. This time I have a question. In regard to the introduction to the computer use, it seems like the graphics part is very often used. For example, in Japan with respect to degree 3 or degree 4 or higher degree polynomials, functions won't be introduced until differential or integral calculus is introduced and, therefore, those graphs are not really introduced because it is very hard to write them manually. It's very complicated. However, it seems that in the States you're using computers a lot and instead of the student drawing graph you let the computer do the work. Now, certainly even in Japan we should be able to introduce this kind of graphics at lower grade levels too. I understand from Professor Fey's demonstration that joined graphs are introduced in the class, but my question is, is this used in general? Is it very popular to do so using computers to draw graphs?

Demana: I think it's very popular and growing now in our country to utilize the technology to build understanding about graphs. In fact, one of the strong suits we see in this use is that students come away with a complete understanding, well almost complete, complete but still at a naive level of understanding of classes of functions. This is the first time in my life I've seen students prior to the study of calculus truly come to grips with what rational functions are. And even before in our country, at least in calculus, we did not see that kind of understanding. And it is growing because it allows more complicated problems for us because the models can be more realistic now that they understand wider classes of functions that can be used as models of problems. I mean we've been able to look at some of the neat applications of conics -

such things as kidney stone crushers which are in parabolic curves and things of that type which now come alive with technology that were not possible before.

Fey: It seems to me that there are two central questions that have come up. One issue is the continuing question of whether mathematics is a product, a body of knowledge that has accumulated over a long period of time and is to be transmitted to students, ora way of thinking, a process? How has technology caused us to re-think our position on this issue? The second issue is the evolving roles of teachers and the student in this new arena. One thing that has struck me about using technology in my own classes is that when I take a computer into the classroom I think the rules of the classroom change. I have come to feel that in traditional mathematics instruction many students feel that mathematics is a secret game which those of us in the fraternity know how to play. They see mathematics education as a contest: It's the students' task to uncover the rules of the game that they're playing against the teacher. But when I bring technology into the classroom, the contest changes. The teacher and the students are now working together to figure out the way the world works. What happens in the classroom is that the relationship between teachers and students changes as the activity changes.

Becker: I feel that the Seminar has been a very valuable experience for all of us and in one way or another has changed each of us, judging by comments made to me by both Japanese and U.S. participants. And I think that what we had hoped for is turning out to be true; namely, that we met here in July of 1986 and were concerned with educational matters, in particular problem solving in mathematics in all its aspects. Following that we organized and carried out cross-national research and other collaborative activities which has now led in a very natural way to this Seminar. I think these things have moved along in a very nice way. We have mathematics educators in our two countries communicating and collaborating with each other which I think is very healthy and useful. Also, I think we have seen some magnificent creations of software here at this Seminar, highly sophisticated software. Further, we have seen software that is very close to the classroom - to the curriculum, to the teacher teaching the content of the curriculum, and to the students. For me, I feel even more sensitive than before to the need to carefully ferret out the implications of some of the more sophisticated software for use by teachers and students in the classroom. And as others have commented, that involves a very large consideration of teacher education. Now with Professor Miwa's concurrence, I think we should change the direction of our discussion towards the Proceedings.

Publication of the Proceedings

Miwa: So, as we have seen, there are many opinions concerning the Seminar deliberations, but now we'll switch to a discussion of the Proceedings of the Seminar.

Becker: We've had no formal discussion of whether a Proceedings should be published or not. There has been discussion during the week that if we decided we would like to do this, then we need to consider what all is involved in that. Since all papers are in English, if we publish the Proceedings it would be in English and, therefore, probably most of the work of putting the Proceedings together would rest with the U.S. side. It would involve editing papers and, of course, each of us re-working our papers following the Seminar in ways that we think appropriate. We would also have to look at the different possibilities in terms of getting the Proceedings printed and whether we might explore, for example, having the National Council of Teachers of Mathematics publish the Proceedings, or possibly contact a commercial publisher. Or, we might consider proceeding as we did in 1986 and print the Proceedings and fairly quickly disseminate them directly to members of the U.S. math education community and also in Japan. The Japanese have indicated that they would not translate all of this into Japanese, but they would be interested in having a substantial number of copies, to be specified later, of the Proceedings for dissemination in Japan. So, are there some of you that have some sentiments you would like to express about these matters?

J. Wilson: I would like to see the Proceedings published, but in a format and in a way which is expedient, to get things out soon rather than the kind of long-term development that might be required with, say, a publisher or with NCTM. So, I would urge that we consider something maybe similar to what we did last time where the goal would be getting it done in an expeditious manner rather than something that if we go through NCTM, for example, we would be talking about two years from now before it would be out.

Becker: Are there other views on the question? The way in which we proceeded before was that Professor Miwa and I collected all the papers, did some light editing of them, had them all typed up on a word processor, proofed and then they were printed at

Southern Illinois University's Duplicating and Printing division. It was done very quickly and then we simply mailed them out from SIUC. We were fortunate at that time to have just barely sufficient resources to send copies to a fairly large number of people in the U.S., and the Japanese helped with resources in disseminating them also.

Demana: I agree with doing this in an expeditious way and we could practice our technological skills and try to take the electronic copies we have and avoid even massive retyping. It would be an interesting scenario to try to pull all of this together electronically.

Becker: Sharon, would you like to make some comments?

Dugdale: I was just thinking that getting it together electronically would be a good idea. But I don't know that making it available on a disk would make sense, given the incompatibility of people's hardware and word processing systems and the non-transportability of some graphic images. So I would suggest that everyone who has an electronic copy should submit it in the most compatible form they can, and that the Proceedings then be compiled as a printed document, rather than a disk.

Teague: I agree with Sharon. I think the problems of having a disk and not knowing how to bring out whatever's on it would be really troublesome.

Wilson: I don't play down the hardware problems and compatibility, but I think the use of a disk has a human side to it - in our geometry and measurement project, we distributed about 800 copies on disk. And we haven't found people who would take the trouble to print them out, and they get very little use; whereas if a hard copy is in hand, they will look at it and make use of some of it. You know that's not a problem having the hardware to do it, the Macintosh is sitting there. In our case, we had (Joe, was it 600-700 pages?) of stuff or something like that? And our production process was on a disk, so then the final copy came from that. So we could make available the disk or the hard copy. It was $3 to do a disk and $25 to do a big notebook, so it was an economic thing.

Becker: Are we talking about two different things? I thought we were talking about the papers being submitted to form the Proceedings.

Dugdale: No, there was a suggestion for an electronic distribution of the Proceedings.

Becker: Okay. What did we decide about the first one? Getting the papers to me to form the volume?

H. Wilson: They can be on electronic form. Whoever gets it has to find a way of getting it off or retyping or whatever? I'm sure you have both MS DOS and Macintosh frames at your place if you're the one that's doing it.

Becker: If I'm the one that's doing it, Macintosh software would be preferred. The woman who would be working on it works very effectively on the Mac.

H. Wilson: With the exception of some problems with graphics, the Macintosh would do. For instance, if it's Word Perfect, it can be carried over with no trouble, with some problems with the graphics.

? Haig But the Japanese computers are completely incompatible with both.

Becker: So perhaps we're speaking only about the U.S. group. From the Japanese, we would like to have the papers and then we would re-type them. Professor Uetake?

Uetake: Sending from Japan, probably the fastest means available now is by fax. However, if sent by fax the question is who is going to pick up the phone bill?

Becker: The person who sends it?

Miwa: And what about the cost for printing?

Becker: Okay. The question is the cost. Mr. Sawada?

Sawada: As far as submission on a Macintosh disk, it's probably not all that difficult since more and more Japanese universities are getting Macintoshes. We could put them in and send the disk. The problem is what to do about the software, not the document itself, but the software?

Becker: Yes, are there comments on this, Sharon?

Dugdale: Even if the Japanese machines are not IBM or Macintosh, couldn't they send a generic text file?

J. Wilson: If they have a Macintosh, they could put the stuff on the Macintosh.

Nohda: Proofing the printing itself is no problem. Well, my concern is whether this is just a collection of the presentations or whether the content for the Proceedings should be more selective and not everything included.

Hashimoto: Since the discussions and various comments are the result of this Seminar, we should include these too - the contents of discussions as well.

Sugiyama: The problem here is not the content of the Proceedings, because the Proceedings should include all the presentations and preferably the contents of the discussion as well. Now the problem is how we should proceed to prepare the material from the Japanese side.

Becker: It's about time for us to have a break. We'll take a break and when we come back we can finish this matter. We need to take a break now for a different reason also; Jim McMahon will be back in a short while and we want to have him and all of his staff in here to thank them for all that they've done for us, before Jim leaves again. So, we'll have a break and then reconvene to take care of that, draw all this together, and then discuss further collaboration.

-----------------------Short Break -------------------------

Becker: We may have to prepare some visuals to give some meaning to the discussions, but we'll work on that. The question of money for printing has been asked and we don't know about that. What we do know is that I can raise some money with NSF, a very small amount in the present grant and maybe some supplemental funding, but certainly not a great deal. And Professor Miwa has said that the Japanese side would look into the possibility of generating some funds which they would use to support the process also. We'll just have to play that by ear, for now, and hope that it works out fine. But I think we need to set a deadline for people getting their disks and/or a hard copy paper to us. Over the break we were talking and the Japanese side would be able to submit all of their papers in MacWrite. How does that sound? Is that possible for the people on the U.S. side also?

Teague: Mine is done on the IBM and I'm not all that anxious to redo it in MacWrite.

Becker: And if that fails, just send us a hard copy and we'll type it into the word processor.

? Can you transport from Word into MacWrite?

Becker: Either, it doesn't matter. Actually the question came up who to send it to. I think I would prefer that it would be sent directly to me and we'll deal with it as best we can. And Jim has indicated that if we need some help on that, he can provide that help. This approach might be just a little bit more efficient in terms of time. Now, what about the deadline for getting these disks and/or hard copy papers to me? What would be a reasonable time to set? Do you have some suggestions?

Hashimoto: I feel that if transcriptions from the discussions are also included, that might take considerable time and so the deadline may go along with that.

Becker: Yes. Maybe if we could set a deadline for getting the papers to me, that will be useful. Of course, we will work as diligently as we can after I get back to my university on getting the transcriptions done. Once those are drafted, a copy would be sent to everyone who has spoken in the discussions. If everybody would like to have a copy of the draft we could provide it. But everyone should see in the transcription what they've said so that if there are any corrections necessary, those could be made.

J. Wilson: If you don't get a response within, say, ten days, you assume they agree with it.

Becker: Well, perhaps. Actually, this didn't turn out to be a problem last time around. I hope it won't be this time either. Can someone suggest a date by which you have your papers to me? I was thinking more like the end of September.

Miwa: I would suggest the end of October.

Becker: That's fine. And then Professor Miwa and I will do some minor editing. If there are any changes made, we'll send you a copy so you're aware of it and you can react to it. Now, the question was asked earlier about software packages, or the software that was demonstrated here, what to do about that? We wonder whether, on each side, we could find someone to provide a description of the software that was demonstrated and that description would be included in the Proceedings. Does that seem satisfactory?

Dugdale: Yes. For the software descriptions, are there any guidelines about length, so we have some notion of consistency?

Becker: That's a good question. Do you have a suggestion?

Dugdale: As far as a suggestion, I think that it could take two pages to include a couple of screen displays and to describe the function of the software. In order to say much more about the interesting problems people were showing it might turn into a fairly large paper, so that's my concern - whether this should be as brief a synopsis as possible or something else.

Miwa: Well, since we don't know the how big the whole Proceedings will be, I'll suggest Professor Becker's approach; that is, the one who demonstrated the software should prepare the description.

Becker: It's all right with me if it's all right with everyone else.

H. Wilson: Jerry. Just a suggestion. It might be helpful if, as early as possible, an appendix could be built with reference numbers or something so that those in the discussion and so forth could reference this appendix list of the software, rather than everybody using different footnotes. I don't know if that's practical or not, but at least there would be a common vocabulary, so to speak, on the software items. Just a suggestion.

Becker: Thank you. Professor Miwa, should we now go to the discussion of the collaboration?

Plans for Post Seminar Collaboration Activities

Miwa: First, I handed out a proposed listing of topics for further study. Though these topics came from discussions I had with the members of the Japanese delegation, it's my own list and has many of my own opinions, so I'm sure that there will be people from the Japanese delegation that have comments and suggestions to add to this. It does not necessarily represent a consensus. I think these are ideas to form joint research, but they're mostly research questions that have come from topics discussed here in the Seminar, and things that I feel need more research. First, teacher education. The importance of teacher education in computer use in school mathematics has been emphasized unanimously in the Seminar. In particular, the content of pre-service education should be studied. Second, software. Excellent software for secondary school mathematics in the U.S. has been demonstrated in this Seminar. Development of software effective for mathematical problem solving using computers should be conducted. We should consider software such as open-end problem solving and game type software. Reference has been made to calculus based on computer use, along with intensive algebra presented by the U.S. in this Seminar. Third, development of a curriculum based upon computer use might be conducted. Four, problems which are appropriate for mathematical problem solving with computer use should also be considered - for example, open-ended problems, appropriate for computer use. Five, teaching using computers should be studied. How the role of the teacher and students' interest change using the computer should be explored. Please give other comments or raise questions.

Demana: Do you mean in two and three that you want the research to be the development, or a survey of what's been developed?

Miwa: Focusing on actual development of software and development of curriculum rather than a survey of what is developed.

Demana: And one other question. Do you include graphing calculators, which are pocket computers, under computers?

Miwa: In #2 I'm focusing primarily on software, rather than hardware. I'm looking at things that either we could develop ourselves or we could push software developers to develop for us.

Sugiyama: When you're thinking about curriculum and use of technology in the classroom, of course, use of the pocket and graphing calculators ought to be within the area that we consider, whether one is actually using them or not.

Fey: I think this is a very impressive list and they're all important problems and there will be work in both countries on these things. What I've puzzled over is what we can do collaboratively. It seems to me one possible strategy is to do case studies or find "existence proofs" in these areas and share those with each other. You have a somewhat different approach to the use of software in the classroom than we do. A rich description of those approaches, videotapes such as the Japanese have shown and things like that that could be shared with the other country, and would be helpful. I don't see research projects in the same sense as the problem solving study, but I think a sharing of the best that you're doing and the best that we're doing would be useful.

Dugdale: Along the same line as Jim's comment, with the #4 problems and #5 teaching and learning, I wonder if something can be done with the process of teaching. This could be somewhat similar to the problem solving study that has already been done, except developing the problems in a teaching and learning situation instead of posing the problems in a test situation. This would involve identifying good problems to develop with computers in the classroom and then perhaps videotaping to compare how the problems are approached and how the students respond in Japanese and U.S. classrooms. But working on the same problems as you identifies in the previous study with the test situation also seems reasonable.

Damarin: I think the focus in two on what's the best software depends in part on what part of the teaching/learning context is being used, so I think one thing we might work on is clarifying the purposes to which software is being put. And I think one interesting approach might be to take some existing software, for example, the Geometer's Sketchpad and look at how that would be most appropriately used in the two countries. What kinds of scripts would be created in different situations and what kind of learning would take place due to the way in which the scripts were created. I'd also like to note that inservice teacher training is not on this list, and it seems to me that it's an important piece of the problem whether that's the study of training itself or the study of how teachers come to be better computer users in each of the countries.

Sawada: Since the our cultures are so different, certainly our environment and the way of teaching are also different. So, at this time of introducing the computer into education, what we would like to, in Japan, introduce individualized teaching and learning situations, whereas it seems like, for example, from the research results of the IEA, the Americans are very much interested in the Japanese way of the whole-class teaching style. If we can exchange those ideas and get the best of each side it would be a very good idea.

J. Wilson: I was going to point out that one of the papers that was distributed to you was something called a "Memorandum" that, in the planning activities on the U.S. side, I agreed to send Jerry a list of activities that might relate to teacher education and the use of computers. Dave Barnes and I put this outline together, and it covers about 12 items, but it covers many of the topics that have been discussed here, sometimes not in papers but in the discussions. But I would argue that there is for each of these kinds of things a research dimension certainly for U.S. classrooms and I suspect that there's a parallel in Japanese classrooms. I echo Jim Fey's concern that if we take on such a broad range of things, that we might have our efforts dissipated, whereas we ought to be dealing with something that is manageable and that can have payoff for us without trying to do everything. And I think that the exchange of information on exemplary practices is one focus to take a look at.

Demana: I'd like to throw out a possibility and let me give a little background. The Second International Mathematics Study is a bit old but I still think there's some stuff in there that's relevant. And I've been particularly happy to be here this week and to see these differences open my eyes, and I'm very thankful to our Japanese colleagues for that. We do have a different point of view and I wonder if we were to take as a basis some of those items and, perhaps adding others, not with an eye of comparing countries again but comparing differences within our countries with classrooms that might be embarking on these technologically rich experiences. And this would give us information within our own country and have a base line from old data.

Becker: Since there are no other comments now, I would like to make a couple comments. I've not discussed the first one with Professor Miwa, Mr. Sawada, Professor Sugiyama or their colleagues yet. Maybe one way to approach possible collaboration would be to get better acquainted with what the classroom situations are in the two countries - the software that's being used and how it's being used. Perhaps an exchange of visits might be one good possibility to get each side more familiar with the situation on the other side. It might also, from the point of view of seeking funding, be a little bit more realistic as a starting point. That's one point. The next point has been touched on, but we might consider some teaching experiments in which some particular content of the curriculum is identified, the planning of the teaching organized, and the selection of the software that would be used to enhance teaching/learning, with some associated evaluation of the effectiveness of the lesson or lessons and the role of the software in the lessons. The next point concerns videotaping. Perhaps we could tape exemplary mathematics lessons in each country in which software is an important aspect of the teaching approach, both for the purpose of studying it and as possible models for use in teacher education programs. Also, examining the teacher education programs in each country might be very useful, and maybe that's something that, if an exchange of visits were to be arranged, could be very useful, Here we would look particularly at problem solving and the use of technology in developing mathematical thinking abilities. Finally, it might be useful to look at all the different ways in which computers are used in classrooms in the two countries, not excluding the extent to which students learn to program in a language and then use the computer in solving certain kinds of problems which maybe are not solveable without the use of that technology. Our time is now getting short, and I'm not sure what to suggest with respect to how we sort through all these various ideas and then select those that we think might be the best on which to focus.

Demana: There's one thing I really feel we didn't bring out. I hope as I hear the terms software and computer use that, to me, graphing calculators are software function graphing utilities in the same sense that the math exploration tool kit and other things are. And given that the Standards in our country assume the use of the graphing calculator for all students in grade 9-12, if in this collaboration the use of word software and computer comes to exclude graphing calculators, it will have made a horrible mistake.

Becker: Then, I would suggest the following agenda for the months ahead. First and foremost, of course, work on the Proceedings. Then on both sides I think we have to look at the possibility of getting some support for the cross-national collaboration whatever its nature turns out to be. And as I've said, I think one first step might be to see if we could work out an exchange of visits, but I've not discussed that with Professor Miwa or his colleagues or anyone else and there might be some reactions to that or some better suggestions. Do you have some comments to make regarding this?

Miwa: I strongly agree with the first proposal that we do need to prepare the Proceedings and we should do this as soon as possible. The next one is that we proposed some topics and some of them we can work on together (e.g., make a survey together) and some of them we can make a survey in each country, and then we're hoping that we'll have something like this seminar again in the near future so that we can meet face-to-face and discuss directly and report and discuss the findings; doing it this way, we can get a lot more out of it than just written information. And, of course, the exchange visit also stimulates and promotes the understanding so we should like to do this also. Exchanging the products that we get out of manufacturing software and stuff like that, by exchanging those ideas as well, will give us a lot of good openings which will stimulate and help to develop better software. I think this should be done too. But in order to do all those things, certainly we do need some support, and so each country should work hard to generate the funding. We have many suggestions generated by all of us and we'll discuss the selection of topics again later, to find out what we really want to do.

Becker: We've approached the time when we need to close the Seminar. So we'll finish now with a brief closing ceremony.

CLOSING CEREMONY

Sawada: In closing, I'd like to thank Professors Becker and Miwa for the success of this conference and also wish their continued good health. I also want to thank Mr. McMahon and his staff for their assistance. We have a small gift for each of you from Japan. I'd like to present these now, as a token of friendship and my wish for continued communication and collaboration. Thank you again, very much.

Miwa: So, now we finish with this closing ceremony. Before we go, Professor Becker and I want to remind you of our reservation at the restaurant, for our Farewell Dinner. We'll meet there at 6:15 p.m.

Fey: Before we are done here, I think we should express our appreciation to Professors Miwa and Becker, who deserve a special round of applause.

REVISED PROPOSED TOPICS FOR FURTHER STUDY

Seminar participants discussed the following topics as a first step towards further collaborative study and research:

1. Teacher Education

   The importance of teacher education in computer use in school mathematics was emphasized unanimously in this seminar. In particular, the content and method of pre-service education should be addressed.

2. Software

   Excellent software for school mathematics has been demonstrated in this seminar. A need for inquiry-oriented and open-ended problem solving software and how it should be used in teaching in the school curriculums of each country was clearly seen. Exchange of information and further discussion of software is recommended.

3. Curriculum

   Strong interest in curriculums based on computer use (e.g., Fey and Heid (1991)) has been expressed. Further study of the potential of this aspect of computer use is recommended.

4. Mathematics Problems

   Problems which are appropriate for mathematical problem solving using the computer should be developed and collected. These problems are to be given to students and evaluated, and they need to be "teacher friendly." Open-ended problems appropriate for computer use should especially be considered.

5. Teaching and Learning

   The role of the teacher and students' interest and attitude changes in connection with computer use has been identified. Furthermore, computer use should be clarified in terms of the aims of teaching mathematics in the classroom.

6. Teaching Units (TUs)

   There was strong interest in development of Teaching Units on mathematical problem solving using computers on each side, trying them out, and exchanging them for tryout and use on the other side, with further revision, if necessary.

DESCRIPTIONS OF SOME SOFTWARE

Software Title: ***Geometry Grapher***
Author: **Jonathan Choate**
Publisher: **Houghton Mifflin Company, Boston, MA**
Donated by: **Houghton Mifflin Company**

See Using Construction Programs in the Teaching of Geometry in this volume for a description of this piece of software and how it is used in the teaching of geometry.

Description by Jonathan Choate

MathCAD

MathCAD is a multi-purpose, flexible and extendible software package for learning and doing mathematics. It combines a simple word processor, a grapher for both functions and data, and a powerful numeric package for evaluating expressions and solving equations. This combination allows students to perform the calculations, write a description of the problem and how their solution works, and illustrate the problem graphically and with tables all in one document. All of the expressions, equations, plots, and tables are live and interactive. By changing parameters in the document, students can see immediately the new results. Versions of MathCAD run on Macintosh, PC DOS, and PC Windows machines. MathCAD 3.1, Windows Version is a significant upgrade from Version 2.5. In addition to operating through the Windows interface, the software has greatly enhanced its visual presentation of mathematical formulas. The software has also added an analytic component to the numerical package with a small set of basic symbolic tools embedded in the software. This symbolic component in MathCAD is based on the Maple symbolic engine developed by Waterloo Maple Software, Inc. The student can study any problem situation graphically, numerically, and analytically.

Our students work with the Student Edition of MathCAD, version 2.0. This reduced set offers a two-page worksheet that has been adequate for most of our student's projects. When the software boots, the student is presented with a blank screen. What happens next is entirely under the student's control. Because MathCAD is so powerful and free-form; it is not a simple matter to learn to use it effectively. In our calculus classes, we have found it beneficial, initially, to set up interactive templates for students to enter and alter. In altering the templates, the students learn to move around in the software and become comfortable with the different aspects of working with data, graphing functions, linearizing data and fitting curves, generating functions through iteration, and solving equations using Newton's Method and differential equations using Euler's Method. As the course progress, the students become responsible for creating more of each document. By the end of the course, they have a tool which is available to them at all times.

MathCAD is a product of MathSoft Inc., 201 Broadway, Cambridge, Massachusetts, 02139.

Description by Dan Teague

Software Title: ***STELLA***

Publisher: High Performance Systems, Lyme, NH

Donated by: High Performance Systems

STELLA is an iconic modeling package which takes full advantage of the Macintosh's mouse and pull-down menu interface. STELLA is an acronym for Structured Thinking Educational Learning Laboratory with Animation. It is an iconic version of the language DYNAMO (DYNAmic MOdels) developed by Jay Forrester of the Sloan School at the Massachusetts Institute of Technology. It was originally designed as a tool for building economic and business simulation models using system dynamics techniques. STELLA is much easier to use than DYNAMO and makes it possible for someone with little training to build dynamic models of systems. All STELLA models consist of tanks and valves or stocks and flows. Tanks are things which accumulate and valves are things which control flows into tanks. Here is how you would use STELLA to build a model of a savings account with an initial balance of $1000 which paid 10% interest compounded quarterly for a period of 10 years starting in 1992.

-Using the mouse select a tank from the tool palette on the left. Drag it onto the screen and label it BALANCE. Double click on the tank and a dialogue window opens up which asks for the initial value of BALANCE. Enter 1000 and close the window.

-Select a valve icon from the tool palette on the left, drag it on to the screen, place it so it sticks into the tank and label it INTEREST.

-Select an arrow icon from the tool palette, and place so it starts at the tank and enters the valve. This passes the current value of BALANCE into the INTEREST valve. Figure 1. shows what the screen would like at this point.

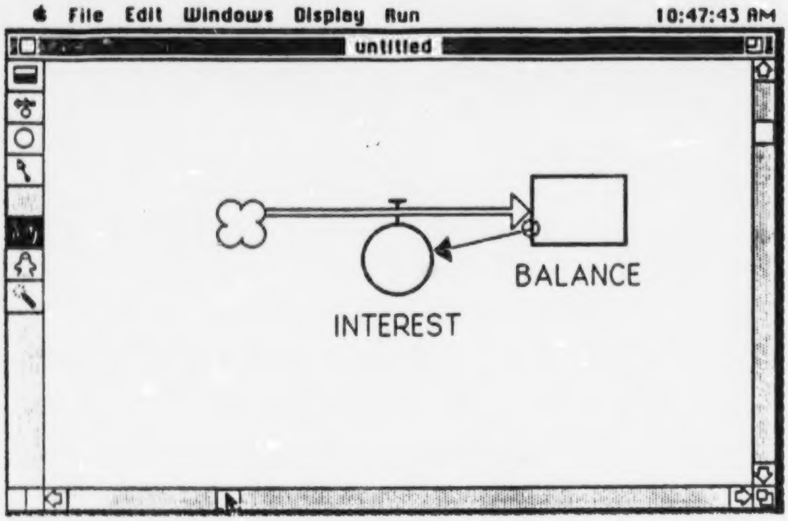
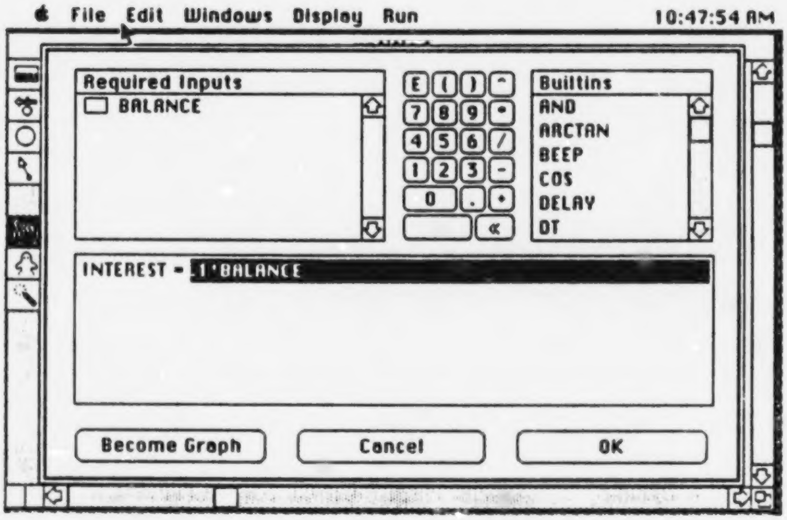

File Edit Windows Display Run
10:47:43 AM
untitled
BALANCE
INTEREST

Fig. 1. STELLA bank balance model

-Double click on the INTEREST valve and a dialogue window like the one shown in Fig. 2. opens up asking you for an equation defining INTEREST and telling you that the equation must contain BALANCE. The arrow from BALANCE to INTEREST requires you to use BALANCE in defining INTEREST.


File Edit Windows Display Run
10:47:54 AM
Required Inputs
BALANCE
E ( ) ^
7 8 9 *
4 5 6 /
1 2 3 -
0 . +
«
Builtins
AND
ARCTAN
BEEP
COS
DELAY
DT
Become Graph
Cancel
OK

INTEREST = .1*BALANCE

Fig. 2. Defining the INTEREST value.

-Go to the WINDOWS menu and select GRAPH. A set of axes with no labels and scales will appear. Double click on this and a dialogue window will open up asking for what you want to graph and what scale you want to use. Enter BALANCE and a vertical scale of 0 to 5000 and close the window.

-Go to the RUN menu and select TIME SPECS. Enter 1992 for the START TIME, 2002 for the FINAL TIME and .25 for DT. Entering .25 for DT is how you tell the program to calculate the interest every quarter of a year.

-The model is now ready to run. Go to the WINDOWS menu, select GRAPH and the set of axes you defined earlier will return. Go to the RUN menu and select RUN. The model is now running and, in a few seconds, a graph similar to the one shown in Fig. 3. will be drawn.

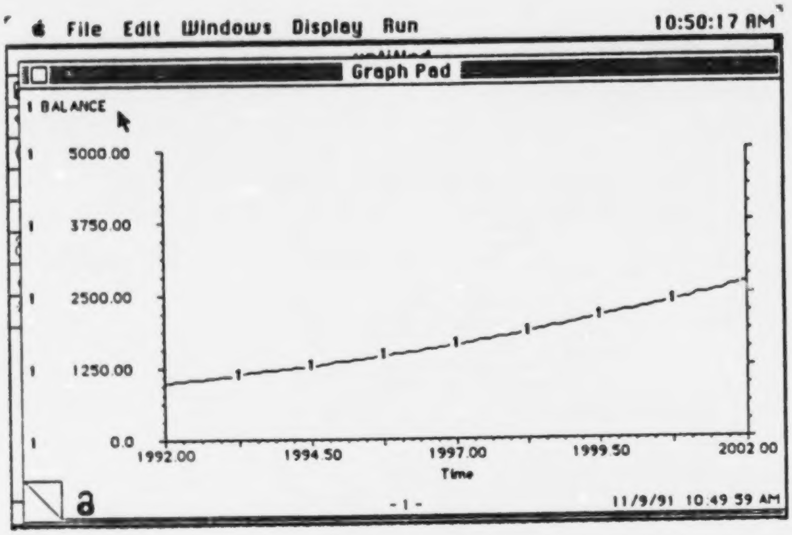

File
Edit
Windows
Display
Run
10:50:17 AM
Graph Pad
1 BALANCE
1 5000.00
1 3750.00
1 2500.00
1 1250.00
1 0.0
1992.00
1994.50
1997.00
1999.50
2002.00
Time
a
- 1 -
11/9/91 10:49 59 AM

Fig. 3. Graph produced by bank balance model

Once the graph is drawn, it can be saved. One can change the model by either entering a new initial value or changing the value for the interest rate. STELLA will also produce a table of values along with the graph.

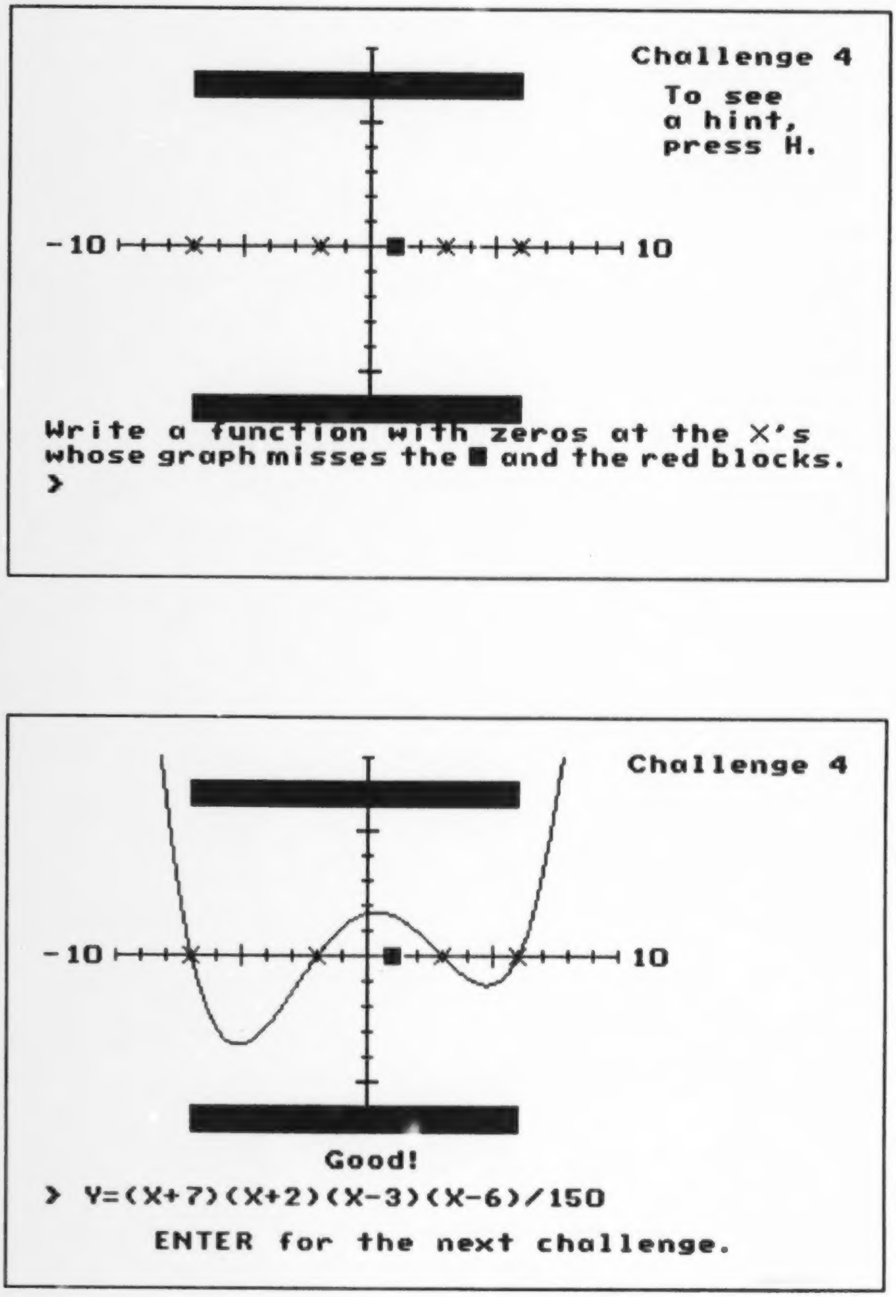
STELLA makes it very easy to build dynamic models of systems for which the rate equations can be defined. This allows students with no knowledge of calculus to build simulation models. Since it is so easy to change a model once it is built, it allows the student to play with a model once it is constructed. STELLA's ease of use and power enables students to easily model a variety of systems and to study their behavior. For this reason, it should be looked at very carefully by all concerned with secondary mathematics education.

Description by Jonathan Choate

Title: ***Slalom, ZOT, ZigZag: Challenges in Graphing Equations***
Authors: **S. Dugdale, D. Kibbey, L. J. Wagner**
Publisher: **Sunburst Communications, Pleasantville, New York**
Donated by: **Sunburst**

This software package provides an informal, graphical introduction to zeros of functions, rational functions, absolute value functions, and pairs of parallel and perpendicular lines. Carefully-sequenced challenges help students develop algebraic techniques and use them graphically. Hints available with the challenges encourage a variety of approaches, such as transformation, addition, and composition of functions.

The structure of the challenges reverses the usual textbook approach of introducing techniques and then providing problems for practice. Instead, a problem is posed, a technique must be devised, and hints (not solutions) are available upon request. The hints offer thought-provoking suggestions and encourage students to develop problem-solving strategies. The screen layout for each challenge is computer-generated to provide variety within specific constraints. Students trying the same challenge several times encounter several different arrangements and need to construct different equations for them. Figure 1 shows a challenge from the Zeros of Functions section.



Challenge 4

To see
a hint,
press H.

Write a function with zeros at the X's
whose graph misses the ■ and the red blocks.
>

Challenge 4

Good!

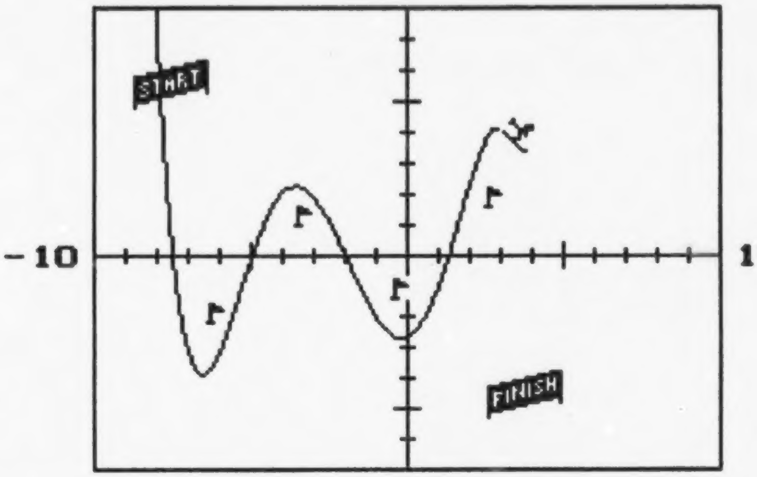
> Y=(X+7)(X+2)(X-3)(X-6)/150

ENTER for the next challenge.

Figure 1. This challenge requires a function that has given zeros and whose graph stays within a limited range between the zeros. In the second frame, the challenge has been successfully completed by first constructing a function that has the requested zeros, then transforming the function to contract the graph vertically so that it stays within the specified bounds.

The techniques addressed in this software draw upon those that students have devised and found rewarding in playing *Green Globs*. (See the software description on page ___.) *Slalom, ZOT, ZigZag* provides a structured environment to encourage *more* students to participate in the sort of creative exploration of functions that some students have initiated in *Green Globs*. The techniques devised by students through experimentation differ somewhat from the usual textbook approaches. For example, students' development of rational functions in *Green Globs* resulted in a basic function to define the overall shape of the graph, plus several individual rational terms, each used to control the behavior of the function near a particular discontinuity. Students' construction of polynomial functions in factored form facilitates manipulation of the zeros and extrema more directly than the conventional expression of polynomial functions as a sum of terms.

Three motivating games provide interactive environments for applying the techniques learned through the challenges. In *Slalom* students construct functions with appropriate zeros, maxima, and minima to guide a skier's path through flags or gates on a ski slope. When an equation is entered, a skier proceeds along the specified path, leaving the graph as a trail, as shown in Figure 2. When students choose gates instead of flags on the ski slope, the skier must go *through* the gates (not just above and below them), so more precision is necessary.



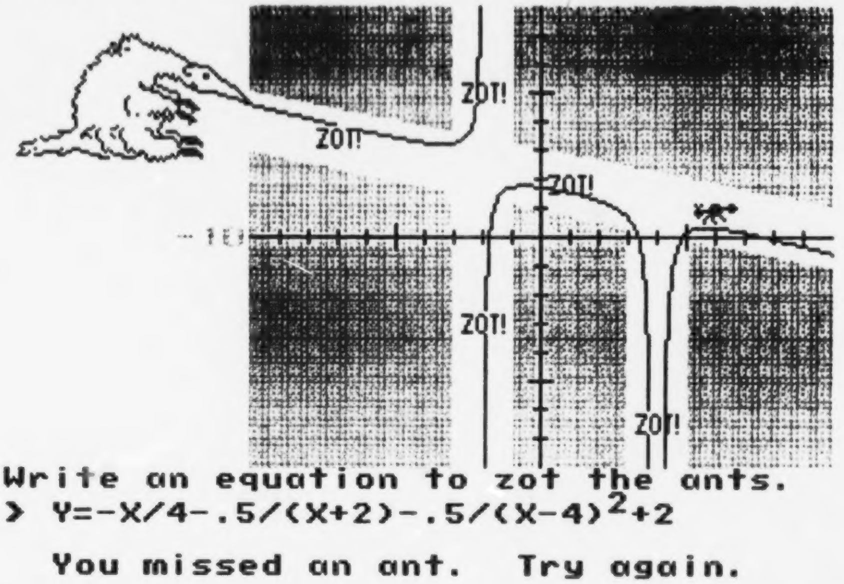
```
> Y=-(X+7.5)(X+5)(X+2)(X-1.3)(X-3.8)/140

calculating
```

Figure 2. The skier clears the fourth flag in a successful game of *Slalom*. The skier must enter through the *START* banner, go alternately below and above the flags, and exit through the *FINISH* banner, without leaving the course bounds before the run is complete.

The challenges in the Rational Functions section are followed by a game, *ZOT!*, in which students combine various techniques to navigate a system of tunnels. Each tunnel pattern is a horizontal or diagonal path with vertical side tunnels. The vertical tunnels may go up only, down only, or both ways. Some intersections may have diagonal obstructions, so that the direction of the graph branches must be planned carefully.

Students playing *ZOT!* may choose to have the targets in the tunnels be ants or abstract rectangular objects. If ants are chosen, the main character of the game is the anteater with the incredible tongue. (See Figure 3.) The anteater's tongue is capable of leaving the screen, wrapping all the way around infinity (that's incredible!), and reentering the screen from the other direction. However, it must avoid any blocks in the tunnels, and it cannot go through the ground around the tunnels.



```
Write an equation to zot the ants.
> Y=-X/4-.5/(X+2)-.5/(X-4)²+2

  You missed an ant.  Try again.
```

Figure 3. A game of *ZOT!* with ants as targets. The anteater has hit five ants, but has missed the sixth. This example requires a function with a linear term and two rational terms. Further, in order to stay inside the tunnels, the second rational term must have negative function values on both sides of the asymptote, so the denominator of this term has been squared and a negative coefficient has been used.

The challenges in the Absolute Value section of *Slalom, ZOT, ZigZag* are followed by a game, *ZigZag*, in which students use absolute value functions to hit targets scattered in a maze of diamond-shaped blocks.

Review by Sharon Dugdale

Software Title: ***Green Globs and Graphing Equations***
Authors: **S. Dugdale & D. Kibbey**
Publisher: **Sunburst Communications, Pleasantville, New York**
Donated by: **Sunburst**

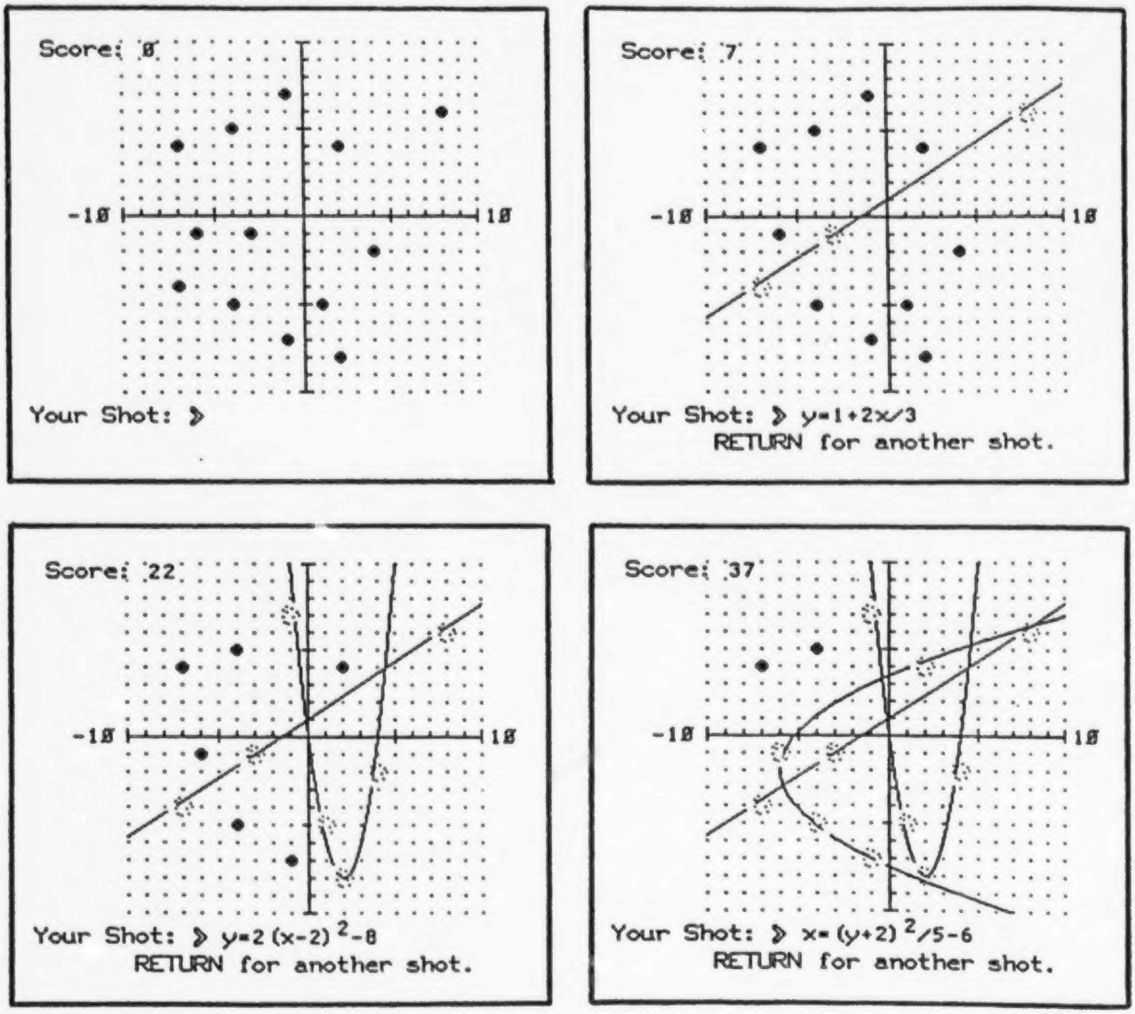
This software package provides mathematical environments for students to manipulate and explore. The content is equations and graphs. The four programs in the package are described briefly below. More detailed discussion of the programs and their use can be found in the software manual and in the references listed below.

Program 1: *Equation Plotter* is a utility program that plots the graphs of entered equations. It is effective for a variety of exploration and problem solving activities. Besides plotting functions beginning with "$y =$" and "$x =$," this program and the others in the package handle conic equations, such as circles, ellipses, and hyperbolas. Functions may include square root, absolute value, logarithmic, exponential, and trigonometric expressions. Eligible expressions are accepted in natural formats. For example, the program does not require "$3*x$," but accepts the usual "$3x$." Likewise it accepts "$\tan^2(y - \pi)$" rather than requiring something like $(\tan(y - 3.14))**2$." Axes may be scaled to suit the problem, and the scaling parameters may include π.

Program 2: *Linear and Quadratic Graphs* provides practice relating equations to graphs. The program displays graphs and asks students to enter an appropriate equation for each given graph. Students' equations are graphed, so that students can compare their own graphs with the target graph and edit their equations as needed to match the target graph. The program offers a choice of lines, parabolas, circles, ellipses, hyperbolas, or a mix of these types of graphs. A series of levels guides students from a basic shape, say $y = x^2$, through increasingly complex transformations of the basic graph, including combinations of reflection, stretch/shrink, and translation.

Program 3: *Green Globs* provides students a compelling environment in which to apply and share what they are learning about equations and graphs. The program displays coordinate axes with 13 green globs scattered to appear randomly placed. The object of the game is to hit all of the globs with graphs specified by entering equations. When a glob is hit, it explodes and disappears. When a shot misses the expected targets, the display of the graph gives the student useful diagnostic information for planning the next shot.

The scoring algorithm encourages students to hit as many globs as possible with each shot. For each shot, the first glob hit is worth one point, the second is worth two points, the third is worth four, and so on. For example, a five-glob shot will score 1 + 2 + 4 + 8 + 16, for a total of 31 points. Figure 1 shows a sequence of displays from the game.



```
Your Shot: ›
```

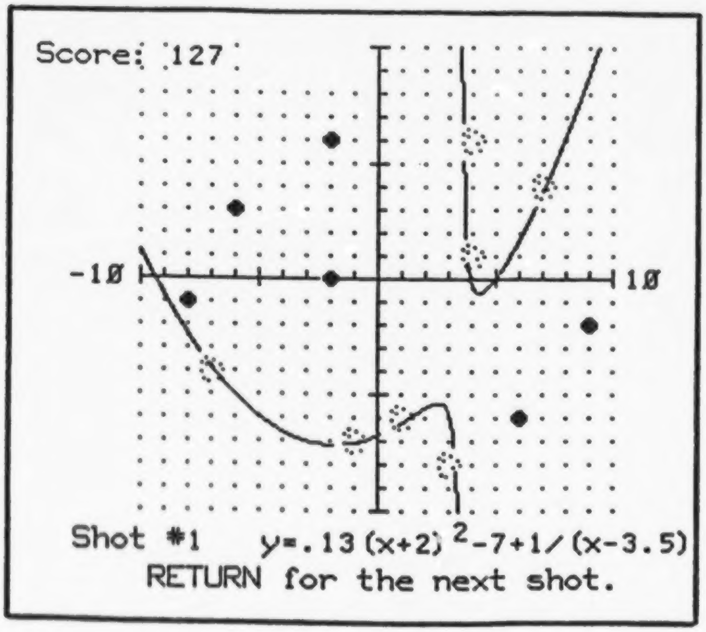
```
Your Shot: › y=1+2x/3
     RETURN for another shot.
```

```
Your Shot: › y=2(x-2)^2-8
     RETURN for another shot.
```

```
Your Shot: › x=(y+2)^2/5-6
     RETURN for another shot.
```

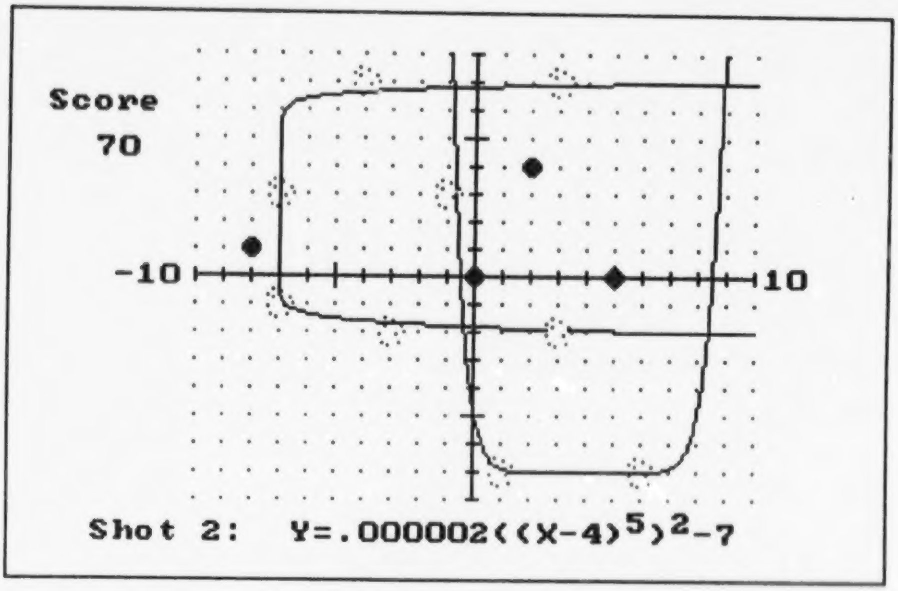
Figure 1. Sequential displays from a game of *Green Globs*. Students enter equations, which are graphed by the computer. The green globs explode as they are hit by the graphs. Shown is the initial display of thirteen globs, followed by a student's first three shots.

The top ten scores are kept in a Records Section, where students' names are displayed along with their record-making scores. Complete information about each game in the Records Section is stored so that the games can be replayed (i.e., viewed as originally played) by other students who want to see what shots and strategies the top-scoring players have used. Sharing of ideas through games stored in the Records Section exposes students to a variety of creative strategies invented by their classmates. Students frequently replay stored games in order to gather new ideas to try in their own games. Examples of the ideas that have appeared in the Records Section are illustrated in Figures 2-4.



```
Shot #1    y=.13(x+2)^2-7+1/(x-3.5)
   RETURN for the next shot.
```

Figure 2. A student has constructed a parabola to hit several green globs, and has then added $1/(x - 3.5)$ to the equation. The effect of this extra term is negligible for all values of x except those close to 3.5, so the resulting graph is nearly the expected parabola except around $x = 3.5$. As x gets close to 3.5, the denominator of the extra term approaches zero, making a vertical asymptote and causing the graph to leave the parabolic path briefly to hit three more globs. Students later extended this technique to create functions with multiple asymptotes.

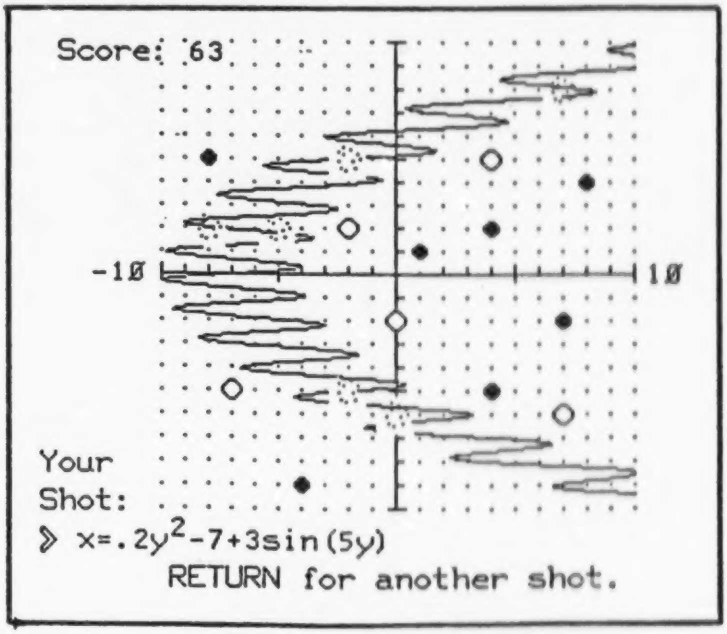


Score

70

Shot 2: Y=.000002((X-4)5)2-7

Figure 3. A student playing *Green Globs* experiments with equations of the form $y = a(x - h)^n + k$, where a is small and n is even.



Score: 63

Your Shot:

> x=.2y2-7+3sin(5y)

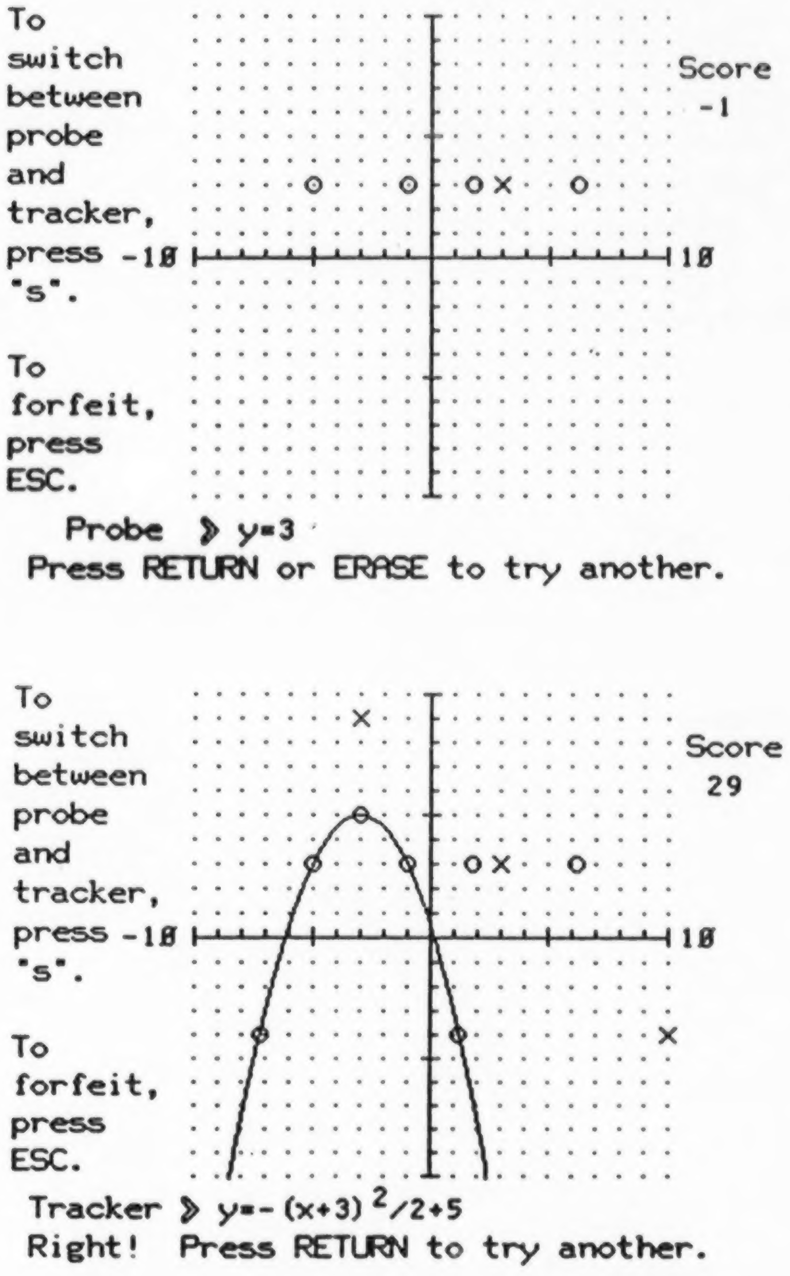
RETURN for another shot.

Figure 4. In the Expert Section of *Green Globs*, students can use trigonometric functions, but their graphs must avoid the five "shot absorbers" scattered among the globs. Here a student has constructed a parabola and then added 3sin(5*y*) to make the graph cover a wider path, while avoiding the shot absorbers.

Program 4: *Tracker* is also a game environment with graphs. Whereas *Green Globs* encourages students to explore many types of functions, *Tracker* deals specifically with linear and quadratic graphs. Students are required to locate graphs that are "hidden" in the coordinate plane and to determine the equations of the hidden graphs. The hidden graphs may be lines, parabolas, circles, ellipses, or hyperbolas, but there are only two types of graphs to find in any one game.

Students input two kinds of "shots" in this game: probes and trackers. Probes are used to find clues about the hidden graphs. Students launch a probe by typing its equation. A probe travels in a horizontal or vertical line, like $x = 4$ or $y = -3$. As a probe goes across the screen, it marks each point where it crosses a hidden graph. Graphs of one type are marked with an 'X,' and graphs of the other type are marked with an 'O.' After gathering enough clues to locate a graph, students send a tracker along the graph by writing its equation. Students can switch freely between shooting probes (to find clues) and trackers (to trace a graph).

The scoring algorithm encourages students to minimize the number of probes and trackers used. Hence, students plan strategies to locate key points for various types of graphs. For example, if the sides of a parabola have been located, a reasonable strategy is to find the vertex by shooting a probe half-way between the sides, along the axis of a symmetry. Or if a probe has crossed a circle, students may need to decide what additional probes are necessary to determine the coordinates of the center. Figure 5 shows two frames from a game of *Tracker*.



```
To
switch
between
probe
and
tracker,
press
"s".

To
forfeit,
press
ESC.

Score
-1

Probe > y=3
Press RETURN or ERASE to try another.
```

```
To
switch
between
probe
and
tracker,
press
"s".

To
forfeit,
press
ESC.

Score
29

Tracker > y=-(x+3) 2/2+5
Right! Press RETURN to try another.
```

Figure 5. Two frames from a game of *Tracker*. In the first frame a probe with the equation $y = 3$ has intercepted hidden graphs at several points. In this student's gave, X's mark points where the probe crossed straight lines, and O's mark points where it crossed parabolas. Using more probes, in the second frame the student has located a parabola. By writing the equation of the hidden graph, the student has sent a tracker along it. There are two more graphs to locate and track.

References

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Dugdale, S., & Kibbey, D. (1984). Some computer applications for the pre-college mathematics and science curriculum. In *Technology in Education and Training: Planning and Management* (pp. 163-174). Silver Spring, MD: Information Dynamics, Inc.

Dugdale, S., & Kibbey, D. (1990). Beyond the evident content goals, part I--Tapping the depth and flow of the educational undercurrent. *Journal of Mathematical Behavior*, 9(3), 201-228.

Review by Sharon Dugdale

SUPPLEMENTARY PAPERS/INFORMATION HANDED OUT AT THE SEMINAR

Henry Jay Becker:	When Powerful Tools Meet Conventional Beliefs and Institutional Constraints: National Survey Finding in Computer Use by American Teachers (September 1990)
Henry Jay Becker:	Mathematics and Science Uses of Computers in American Schools, 1989 - Data and Analyses From the U.S. Participation in the I.E.A. Computers-in-Education Survey (December 1990)
Henry Jay Becker:	How Computers Are Used in United States Schools: Basic Data From the 1989 I.E.A. Computers in Education Survey (November 1990)
Jerry P. Becker:	Cross-National Mathematics Achievement Results and Observations Concerning Problem Solving and Creativity of American and Japanese Students (July 1991)
Jerry P. Becker:	Problems Used in the U.S.-Japan Cross-National Research Grades 4, 6, 8, 11 (April 1991)
Japan Society of Mathematical Education:	Mathematics Program in Japan (January 1990)
Hiroshi Fujita, Tatsuro Miwa, and Jerry P. Becker:	The Reform of Mathematics Education at the Upper Secondary School Level in Japan (1991)
National Institute for Educational Research:	Basic Facts and Figures About the Educational System in Japan (March 1990)
Japanese Delegation:	Japanese School Mathematics Textbooks and Computer Software
Japanese Delegation:	Videotape on Software Use in Japanese Classrooms
U.S. Delegation:	U.S. School Mathematics Computer Software
W. Gary Martin:	Handout on the Hawaii Geometry Learning Project
East-West Center:	Information About the East-West Center